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THE
METHOD
OF
FLUXIONS

L'Hôpital 7 A
BOTH
DIRECT *and* INVERSE.

The FORMER being
A Translation from the Celebrated
Marquis De L'HOSPITAL'S *Analyse*
des Infiniments Petits:

And the LATTER
Supply'd by the TRANSLATOR,

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PREFACE.

THE Analysis explain'd in the following Work supposes the common Analysis, but is of a very different Nature from it; the latter being confin'd to Finite Quantities, whereas the former extends to Infinity itself. By means of this Analysis we compare the infinitely small (Differences or) Parts of finite Magnitudes, and find their Ratio's to each other; and hereby likewise learn the Ratio's of finite Magnitudes, those being in reality so many infinitely great Magnitudes, in respect of the other infinitely small ones. This Analysis may ever be said to go beyond the Bounds of Infinity itself; as not being confined to infinitely small (Differences or) Parts, but discovering the Ratio's of Differences of Differences, or of infinitely small Parts of infinitely small Parts, and even the Ratio's of infinitely small Parts of these again, without End. So that it not only contains the Doctrine of Infinites, but that of an Infinity of Infinites. It is an Analysis of this kind that can alone lead us to the Knowledge of the true Nature and Principles of Curves: For Curves being no o-

ther than Polygons, having an infinite Number of Sides, and their Differences arising altogether from the different Angles which their infinitely small Sides make with each other, it is the Doctrine of Infinites alone that must enable us to determine the Position of these Sides, in order to get the Curvature formed by them; and thence the Tangents, Perpendiculars, Points of Inflexion and Retrogression, reflected and refracted Rays, &c. of the Curves.

Polygons circumscribed about or inscribed in Curves, whose Number of Sides infinitely augmented till at last they coincide with the Curves, have always been taken for the Curves themselves. But the Notion rested here, without farther Improvement, for many Ages: And it was the Discovery of the Analysis of Infinites that first pointed out the vast Extent and Fecundity of this Principle.

What the Ancients, and particularly *Archimedes*, have done herein, may claim our Wonder; but then they have only consider'd a few Curves, and those too, slightly enough: They have left us little other than a Course of particular Propositions, which give no Indication of any uniform and consistent Method. Yet ought they by no means to be reproached on this score, considering the great Force of Genius * they shew'd in penetrating such Obscurities, and setting foot in a Land utterly unknown before: If they did not tra-

* *Though I have twice or thrice read over Archimedes's Treatise of Spirals with the utmost Attention, to comprehend the Art employ'd in his subtle Demonstrations relating to the Tangents of Spirals, yet cou'd I never rise from him without some suspicion that I had not here taken the whole Force of the Demonstration, &c.* BULLIALDUS *Præf. de Lineis Spiralibus.*

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vel far, nor took the most direct Way, yet whatever *Vieta* * may imagine, they did not deviate far; and the more difficult and thorny the Routs they took, the more surprizing it is they were not lost. In a word, the Ancients appear to have gone as far as their Time could admit of, and done the same that our later Genius's would, if under the same Circumstances: And had the Ancients lived in ours, it is reasonable to believe, they would have had our Views; this being a Consequence of the natural Equality we find in Genius's, and the necessary Time required for the Succession of Discoveries. 'Tis therefore no Surprize the Antients have not gone farther in this Affair; but very strange indeed that great Men, undoubtedly as great as any of the Ancients, shou'd so long have sat down here, and with a kind of superstitious Veneration contented themselves to read and comment the Works of Antiquity, without allowing themselves any other Use of their Faculties, than barely serv'd them as Followers, that durst not think for themselves, or carry their Views beyond the Discoveries of their Predecessors.

In this manner many employ'd themselves; they wrote; the Number of Books increased, yet no farther Progress made; the Labours of many Ages serving only to fill the World with obsequious Comments and multiplied Translations of Originals, that in themselves were often worthless. Such was the State of Mathematicks, and principally of Philosophy, till the Time of *Descartes*; who, at the Instigation of his

* *Si verè Archimedes, fallaciter conclusit Euclides, &c.*
Suppl. Geom.

great and commanding Genius, left the Ancients, to follow the same Guide, *Reason*, that had conducted them: And this happy Bravery, tho' treated as a Revolt, was follow'd by an infinite Number of new and useful Views, both in Philosophy and Geometry. People now begun to open their Eyes and think for themselves.

To keep to Mathematicks: *Descartes* here began where the Antients left off, viz. with the Solution of a Problem, at which, according to *Pappus*, [in *Collect. Mathem.* Lib. 7. at the Beginning] all of them stuck. 'Tis well known to what a Pitch he carried Algebra and Geometry, and how easy, by the Introduction of the former into the latter, he has render'd Solutions of innumerable Problems, which no one before him could master. But as he principally applied himself to the Resolutions of Equations, Curves were by him consider'd no farther than as they might assist him in finding their Roots: So that common Algebra being sufficient for this, he did not endeavour to find any other, except in the Business of drawing Tangents to Curves, where he has happily applied it; and the Way he discover'd for that purpose appear'd to him so excellent, that he did not scruple to say, "this Problem * was the most useful and general, not only that he then knew, but even that he ever had a Desire to know in Geometry."

* Geom.
Lib. 2.

The Geometry of *Descartes* having brought the Construction of Problems by the Resolutions of Equations into great vogue, and given a considerable Insight into the Affair, the major part of Geometricians now apply themselves to study and improve it with their own
Disco-

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Discoveries, which thus daily advance it towards Perfection.

Monfieur *Paschal* indeed directed his Views quite another way: He examined Curves in themselves; and under the Form of Polygons, found out the Lengths of some, the Spaces contained under them, the Solids described by those Spaces, their Centres of Gravity, &c. And from a bare Consideration of their Elements, that is, their infinitely small Parts, he discovered some general Methods relating thereto: Which are the more surprizing, as he seems to have come at them without Algebra, by the sole Force of Imagination.

Soon after the Publication of *Descartes's* Method of Tangents, Monsieur *De Fermat* discovered another, which *Descartes* himself at length allows, (*Sett. 71. Tom. 3.*) to be more simple than his own on several Occasions. Yet this itself is not so simple as Dr. *Barrow* afterwards made it, from a close Consideration of the Nature of Polygons, which naturally represent to the Mind a little Triangle consisting of a Particle of a Curve, (contained between two infinitely near Ordinates,) the Difference of the correspondent Absciss's; and this Triangle is similar to that formed by the Ordinate, Tangent, and Subtangent: So that by one simple Analogy, this Method of Dr. *Barrow's* performs the Business, without the Calculus required in the Method of *Descartes* and *De Fermat*.

Dr. *Barrow* rested not here: He also invented a kind of Calculus suitable to the Method, (*Lect. Geom. page 80.*) tho' deficient as well as that of *Descartes*, in clearing Equations of Fractions and Surd Quantities.

• Acta E-
rudit. Lipf.
Ann. 1684.
p. 467.
† See Com-
mercium
Epistoli-
cum.

The Defect of this Method was supplied by that of Mr. *Leibnitz's*, * [or rather the great *Sir Isaac Newton*. †] He began where Dr. *Barrow* and others left off: His *Calculus* has carried him into Countries hitherto unknown; and he has made Discoveries by it astonishing to the greatest Mathematicians of *Europe*. The Messieurs *Bernouli* were the first who perceived the Beauty of the Method; and have carried it such a length, as by its means to surmount Difficulties that were before thought insuperable.

This Calculus is of vast Extent; as being suited to mechanical Curves, as well as geometrical ones. Radical Signs are no Incumbrance at all in it, but sometimes a Convenience. It extends to any number of indeterminate Quantities; and the Comparison of infinitely small Quantities of all kinds is performed by it with equal Facility. Whence arise an Infinity of surprizing Discoveries with regard to Tangents, as well Curves as right Lines, to Problems *de maximis & minimis*, Points of Inflection and Regression, Evolutes, Causticks by Reflection and Refraction, &c. as will appear in the Work itself.

I have divided it into ten Sections; the first whereof contains the Principles of the Method; the second shews the use thereof, in finding the Tangents of all kinds of Curves, let what will be the number of indeterminate Quantities in the Equation expressing their Natures. Tho' Mr. *Craigs*, indeed, (in *Lib. de Quadr. Figurar. Curvilinear. part 2.*) thinks it only applicable to geometrical Curves. The third shews the use of the Method in solving all Problems *de maximis & minimis*. The fourth

fourth determines the Points of Inflexion and Retrogression of Curves. The fifth shews how to find the Evolutes of Monsieur *Hugens* in all kinds of Curves. The sixth and seventh shew the Method of finding Causticks by Reflection and Refraction, whereof M. *Tschirnhausen* is the Inventor. The eighth contains the farther use of the Method in finding Points in Curves that touch an infinite number of Right Lines, or Curves of a given Position. The ninth contains a Solution of some Problems that depend upon the foregoing Discoveries: And the tenth exhibits a new way of using the *Calculus Differentialis*, (or Method of Fluxions) in geometrical Curves: From whence is deduced the Method of *Descartes* and *Hudde*, which is only applicable to such kind of Curves.

In the 2^d, 3^d, 4th, 5th, 6th, 7th, and 8th Sections, there are indeed only a few Propositions: But then they are all General. And, as it were, so many Methods, easily applicable to any number of particular Propositions. But I have only applied them in some select Examples, being persuaded, that in Mathematicks general Methods are best; and that the Books containing Details, or particular Propositions, misemploy the Time both of the Reader and Author. Whence I should not have added the Problems of the 9th Section, if they were not curious and exceedingly general. Thus the 10th Section likewise contains nothing but Methods which the *Calculus Differentialis* gives to the manner of *Descartes* and *Hudde*, for drawing of Tangents. And if these are less general, all the preceeding Methods shew, that the Fault is not in our Calculus

culus, but in the Manner of *Descartes*, where-
to it is confined.

On the other hand, there can be no better
Proof of the vast Use of our Calculus, than
this great Variety of Methods, as compre-
hending the Whole of what *Descartes* and
Hudde have done in the Affair of Tangents.
And the universal Proof it gives us of the use
of arithmetical Progressions therein, leaves no
room to doubt of the Certainty of this last
Method.

I intended to have added another Section,
to shew the surprizing use of this *Calculus* in
Physicks, and to what degree of Exactness it
may bring the same; as likewise the use there-
of in Mechanicks; But Sickness has prevented
me herein. However, I hope to effect it hereaf-
ter, and present it the Publick with Interest. And
indeed the Whole of the present Treatise is on-
ly the First Part of the Calculus of M. *Leib-
nitz*, or the *Direct Method*, wherein we de-
scend from Whole Magnitudes to their infi-
nitely small Parts, of what kind soever. com-
paring them with each other, which is called
the *Calculus Differentialis*: But the other Part,
called the *Calculus Integralis*, (or *Inverse Me-
thod of Fluxions*,) consists in ascending from
these infinitely small Parts to the Magnitudes,
or Wholes, whereof they are the Parts. This
Inverse Method also I designed to publish;
but Mr. *Leibnitz*'s having wrote to me, that he
was at work upon this Subject, in order for a
Treatise de Scientia Infiniti, I was unwilling to
deprive the Publick of so fine a Piece, which
must needs contain whatever is curious in the
Inverse Method of Tangents, Rectifications
of Curves, Quadratures, Investigation of Su-
perficies

perficies of Solids, and their Solidities, Centres of Gravity, &c. Neither would I ever have published the present Treatise, had he not intreated me to it by Letter; as likewise because I believed it might prove a necessary Introduction to whatever shall hereafter be discovered on the Subject.

I must own my self very much obliged to the Labours of Messieurs *Bernoulli*, but particularly to those of the present Professor at *Groenengen*, as having made free with their Discoveries as well as those of Mr. *Leibnitz*: So that whatever they please to claim as their own, I frankly return them.

I must here in justice own, (as Mr. *Leibnitz* himself has done, in *Journal des Sçavans* for *August*, 1694.) that the learned Sir *Isaac Newton* likewise discover'd something like the *Calculus Differentialis*, as appears by his excellent *Principia*, published first in the Year 1687. which almost wholly depends upon the Use of the said *Calculus*. But the Method of Mr. *Leibnitz*'s is much more easy and expeditious, on account of the Notation he uses, not to mention the wonderful Assistance it affords on many Occasions.

When this Treatise was nearly printed off, Mr. *Nieurventii*'s Performance happened to come to hand; the Title whereof being *Analysis Infinitorum*, gave me the Curiosity of running it over; upon which I found it very different from mine: For the Author not only uses a Notation different from Mr. *Leibnitz*'s, but absolutely rejects second, third, &c. Differences (or Fluxions;) and as the greater Part of my Book is built upon that Foundation, I should have thought my self obliged to answer
his

his Objections, and shew their Insufficiency, if Mr. *Leibnitz*'s had not already fully done it to my hands, in the *Acta Eruditorum* An. 1695. p. 310 and 369.

To conclude: The two *Postulata* or Suppositions laid down at the Entrance of this Work as the sole Foundation thereof, appear to me so self-evident, as not to leave the least Scruple about their Truth and Certainty on the Mind of an attentive Reader. They might however have been demonstrated after the Manner of the Antients, if I had not intended to be short in things already known, and apply my self principally to such as are new.





THE
TRANSLATOR
TO THE
READER.

I*t is needless for me to say any thing in Commendation of the Author's Piece, the Character whereof is so well establish'd: And had he wrote likewise a second Part, or the Inverse Method of Fluxions, or Calculus Integralis, (as Foreigners call it) the Whole would, no doubt, have been an excellent Introduction to this admirable Doctrine.*

My Design at first was to have published the Translation alone; but considering that it would be imperfect without the Inverse Method, I therefore have supplied it, and shewn its Uses in the Quadratures of Curv'd-lin'd Spaces, the Rectifications of Curves, Cubations of Solids, Quantities of their Superficies, and the Investigations of Centres of Gravity and Percussion; the Whole concluding with a few miscellaneous Problems.

*The first Section, being a general Introduction, handles the Doctrine of infinite Series, which is so necessary to what follows, that without it, a
Person*

Person can have but a very slender Skill in the Inverse Method of Fluxions.

In the second Section, concerning the finding of Fluents, you have not only the usual Ways of performing the Business, (as well when they cannot be had in finite Terms, as otherwise,) but likewise the Manner of using the Tables of the late learned Mr. Cotes for that end: By which, and the Tables of Logarithms and natural Sines, elegant Expressions of Fluents may be had, without the Labour of throwing Quantities into Series, which in many Cases is not a little embarrassing.

In the succeeding Sections you have a Number of select Examples, whose Solutions will more divert than trouble, and at the same time sufficiently instruct the industrious Learner. Among these you have the analytical Processes of all the Problems of Mr. Cotes, about Quadratures, Rectifications, Cubations, and the Quantities of the Superficies of round Solids, (the bare Constructions whereof he has given us in his admirable Treatise, entituled, *Harmonia Mensurarum*,) from whence we gain the neat and elegant Constructions he has given in the aforesaid Treatise.

In the Section concerning the Investigations of the Centres of Gravity, you will find a Mean observ'd in the Choice of Examples, both as to Number and Facility of Solution: You have here also the Determination of the Centres of Gravity of Hyperbolick and Elliptick Spaces in the Measures of Ratio's and Angles. In the next Section you have nothing new: Those that have a mind to try, may perhaps get not inelegant Constructions of Fluents, determining the Centres of Percussion of Hyperbolick and Elliptick Spaces, &c. after Mr. Cotes's Way.

The first four Problems at the End may serve to give the Learner a Taste of the Inverse Method of Tangents, about which I might have been more full, but these alone will teach the Manner of solving others of the like nature. And it is for this Reason I have given also some physical Problems, which may direct him to master those of a more difficult Kind, whenever they occur.

As the illustrious Author has omitted the Exponential Calculus, or Manner of finding the Fluxions of Exponential Quantities, such as $x^x = a$, $x^x = y^y$, &c. where the Index's of the variable Quantities are also variable; thinking, as I suppose, this Branch of Doctrine to be of very little or no use, so I have been silent in this Matter also; which it is much better to be, than take up the Reader's Time in learning what is only mere Speculation. Thus you have a summary View of what is contained in the Appendix. Once for all; that I may not be thought a Plagiary, I freely own, many things are here taken from Sir Isaac Newton, &c.

In a word, I am of opinion that the Translation, together with the Appendix or second Part, may serve very well for an Introduction to the Doctrine of Fluxions, and will so far qualify the Learner, as to render him capable of understanding with moderate Ease the more sublime Discoveries in Mechanics, Physicks, &c. that have been found by this Art. Such a Work becomes the more necessary, because there are but two English Treatises on the Subject, (that I know of) the one being Hays's Introduction to Mathematical Philosophy, and the other, Ditton's Institution of Fluxions; the former of which is by far too prolix for Learners, abounding in general Rules and Examples, many of which will rather
confound

confound than instruct him; and is at the same time deficient in things that would not a little forward him; not to mention the Book's being out of Print, and very likely for ever so to continue.

On the other hand, Mr. Ditton's Book is by much too sparing in Examples of the Uses of the Methods: Those few he has given us being by no means fit for Beginners. He is also too redundant at his first setting out, in the Explanation of the Definition of Fluxions, as Sir Isaac Newton has it: Which, tho' it be true and exact, it is next to impossible for one who has not been conversant about Infinites to apprehend it. That of our Author is much easier, tho less Geometrical, who calls a Differential (or Fluxion) the infinitely small Part of a Magnitude; not deterring his Readers at first from proceeding, by dwelling long on the Explication of an intricate Definition, but comes immediately to the Algorithm, or Arithmetick of the Art; and thence to plain Examples of Solutions by it.

But I would not here be thought in any wise to lessen the Value of Sir Isaac Newton's Definition: When the Learner has made some Progress, I would have him then make himself Master of it.

Thus much for the Work; preparatory to the due Perusal whereof, it may not be amiss to give some Notion of the general Nature and Origin of Fluxions, according to the Sense of the great Author and Inventor thereof, Sir Isaac Newton.

In order to this, we are to consider Quantities not as made up of very small Parts, but as described by a continued Motion. For Example, a Line is described, not by the Apposition of little Lines or Parts, but by the continual Motion of a Point. A Surface or plain Superficies is descri-
bed

bed by the Motion of a Line, (not according to its own Direction.) And a Solid, by the continual Motion of a Superficies, (not according to its own Direction.)

Now the Velocities of the Increases or Increments of Magnitudes thus moving in very small equal Particles of Time, at the first Instant of the Generation of those Increments are called Fluxions, and those Magnitudes Fluents, or Flowing Quantities.

These Fluxions are nearly proportional to the Increments of the Fluents or Flowing Quantities, generated in very small equal Parts of Time; but accurately as the Velocities wherewith they arise and begin to be generated; that is, they are those very Velocities; as was said before.

The reason why the Increments generated in small equal Parts of Time, are not exactly proportional to the Velocities wherewith they are generated, is, because those Velocities are not constantly equable, which, if they were, the little Spaces described in equal Particles of Time, would then be exactly as the Velocities, (which is an allowed Principle in Mechanicks;) but the Velocities are mutable, or accelerated continually, and so the said little Spaces described, cannot be constantly proportional to the Velocities wherewith they were first described, or proportional to any one of those mutable Velocities.

Yet if the Particles of Time be taken very small, and the Acceleration be so likewise, Fluxions may be taken as proportional to those Increments, and so the said Increments, may represent Fluxions in all Operations; the Results of which will be as exact, as if they had been determined from the Velocities of the Motions wherewith the said Increments begin to be generated. But to scrupulous

Persons the Processes will not be allowed so geometrical. Neither indeed is it so satisfactory to throw out Quantities from an Equation, on account of their being infinitely less than others, (as all who use an Increment for a Fluxion do,) as to reject them, because they really vanish and become equal to nothing, as Sir Isaac Newton does.

*See Com-
mercium
Epistol-
icum.*

Upon this latter Foundation is built the Calculus Differentialis, first published by Mr. Leibnitz, in the Year 1684.; having been since followed by almost all the Foreigners: Who represent the first Increment, or Differential, (as they call it,) by the Letter d, the second by dd, the third by ddd, &c.; the Fluents, or Flowing Quantities, being called Integrals. But since this Method in the Practice thereof, does not differ from that of Fluxions, and an Increment or Differential may be taken for a Fluxion; out of regard to Sir Isaac Newton, who invented the same before the Year 1669, I have altered the Notation of our Author, and instead of d, dd, d³, &c. put his Notation, viz. $\dot{x}$, $\ddot{x}$, $\dddot{x}$, &c. or some other of the final Letters of the Alphabet, pointed thus, and called the infinitely small Increment, or Differential of a Magnitude, the Fluxion of it.*

So far of the general Nature of Fluxions: I shall conclude with proposing the Solutions of the two following Problems to such as are capable.

I. The Latitude of a Place and the Day of the Year being given, to find the Time of that Day the Heat of the Sun shall be the greatest, admitting it be as any Power of the Sun's Duration above the Horizon, and as any Power of the Sine of his Altitude.

II. To

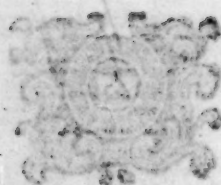
II. To find the Nature of one of the Superficies of a Wall of uniform Matter, of a given Height, Length, Breadth at Bottom (*viz.* the Ground) and Top, the other opposite upright Superficies being a right-angled Parallelogram, as also the Section at the Ground and the Top, that shall stand firmer, or be least disposed to fall, when the Wind blows directly against the upright plain Superficies of it, than when the Superficies (to be found) is of any other Nature; allowing the Particles of the Air to be equally distant from each other, of equal Magnitudes, and all to move with equal Velocities parallel to each other; and the Wall to break off at the Surface of the Ground, with a sufficient Force of Wind.

Note, The three Sides of the infinitely small Triangle, mentioned in Page vij. aforegoing, are an infinitely small Part of the Curve, the Difference between two infinitely near Ordinates, and that of two infinitely near Absciss's.



II. To find the Nature of one of the Superficies of a Wall of uniform Matter, of a given Height, Length, Breadth at Bottom (over the Ground) and P, the other opposite upright Superficies being a right-angled triangle, as also the Section at the Ground and the Top, that shall stand firm, or be least disposed to fall, when the Wind blows directly against the upright plain Superficies of it, than when the Superficies to be found is of any other Nature; allowing the Particles of the Air to be equally distant from each other, of equal Magnitude, and all to move with equal Velocities parallel to each other, and the Wall to break off at the Centre of the Ground, with a horizontal Force of Wind.

Note, The three Sides of the upright plain Surface, mentioned in Page 19, are given, as an instance, from Part of the Curve, the Distance between two upright plain Surfaces, and that of two upright plain Surfaces.





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A TREA-



A
T R E A T I S E
O F
F L U X I O N S.

P A R T I.

S E C T. I.

Of finding the Fluxions of Quantities.

DEFINITION I.



Variable Quantities are those that continually increase or decrease; and *constant* or *standing Quantities*, are those that continue the same while others vary. As the *Ordinates* and *Abscisses* of a Parabola are variable Quantities, but the *Parameter* is a constant or standing Quantity.

DEFIN. II.

THE infinitely small Part whereby a variable Quantity is continually increased or decreased, is called the **Fluxion* of that Quantity.

B

For

* See the Translator's Preface.

FIG. 1. For Example: Let there be any Curve Line AMB , whose Axis or Diameter is the Line AC , and let the right Line PM be an Ordinate, and the right Line pm another infinitely near to the former.

Now if you draw the right Line MR parallel to AC , and the Chords AM , Am ; and about the Centre A with the Distance AM , you describe the small circular Arch MS : then shall Pp be the Fluxion of PA ; Rm the Fluxion of Pm ; Ss the Fluxion of AM ; and Mm the Fluxion of the Arch AM . In like manner, the little Triangle MAm , whose Base is the Arch Mm , shall be the Fluxion of the Segment AM ; and the small Space $MPpm$, will be the Fluxion of the Space contained under the right Lines AP , PM , and the Arch AM .

COROLLARY.

1. **I**T is manifest, that the Fluxion of a constant Quantity, (which is always one of the Initial Letters a, b, c , &c. of the Alphabet) is 0: or (which is all one) that constant Quantities have no Fluxions.

SCHOLIUM.

THE Fluxion of a variable Quantity expressed by a single Letter, which is commonly one of the later Letters of the Alphabet, is represented by the same Letter with a Dot or Full-point over it: as the Fluxion of x is $\dot{x}$, that of y is $\dot{y}$, that of z is $\dot{z}$, and that of u is $\dot{u}$. And if you call the variable Quantities AP, x ; PM, y ; AM, z ; the Arch AM, u ; the mixt-

of FLUXIONS.

3

lin'd Space APM , s ; and the Segment AM , t : then will x express the Value of Pp , y the Value of Rm , z the Value of Sm , u the Value of the small Arch Mm , v the Value of the little Space $MPpm$, and i the Value of the small mixt-lin'd Triangle MAm .

POSTULATE I.

2. **G**RANT that two Quantities, whose Difference is an infinitely small Quantity, may be taken (or used) indifferently for each other: or (which is the same thing) that a Quantity, which is increased or decreas'd only by an infinitely small Quantity, may be consider'd as remaining the same.

For Example: Grant that Ap may be taken for AP ; pm for PM ; the Space Apm for APM ; the small Space $MPpm$ for the small Rectangle $MPpR$; the small Sector AMS for the small Triangle AMm ; the Angle pAm for the Angle PAM , &c.

POSTULATE II.

3. **G**RANT that a Curve Line may be consider'd, as the Assemblage of an infinite Number of infinitely small right Lines: or (which is the same thing) as a Polygon of an infinite Number of Sides, each of an infinitely small Length, which determine the Curvature of the Line by the Angles they make with each other.

For Example: Grant that the Part Mm of the Curve, and the Circular Arch MS , may be consider'd as straight Lines, on account of their being infinitely small, so that the little

Triangle mSM may be looked upon as a right-lin'd Triangle.

P R O P. I.

4. **T**O find the Fluxions of simple Quantities connected together with the Signs $+$ and $-$.

It is required to find the Fluxion of $a + x + y - z$. If you suppose x to increase by an infinitely small Part, viz. till it becomes $x + \dot{x}$; then will y become $y + \dot{y}$; and $z, z + \dot{z}$: and the constant Quantity a will * still be the same a . So that the given Quantity $a + x + y - z$ will become $a + x + \dot{x} + y + \dot{y} - z - \dot{z}$; and the Fluxion of it (which will be had in taking it from this last Expression) will be $\dot{x} + \dot{y} - \dot{z}$; and so of others. From whence we have the following

R U L E I.

For finding the Fluxions of simple Quantities connected together with the Signs $+$ and $-$.

Find the Fluxion of each Term of the Quantity proposed; which connected together by the same respective Signs will give another Quantity, which will be the Fluxion of that given.

P R O P. II.

5. **T**O find the Fluxions of the Product of several Quantities multiplied, or drawn into each other.

1. The Fluxion of xy is $y\dot{x} + x\dot{y}$: for y becomes $y + \dot{y}$, when x becomes $x + \dot{x}$; and therefore xy then becomes $xy + y\dot{x} + x\dot{y} + \dot{x}\dot{y}$. Which

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Which is the Product of $x + \dot{x}$ into $y + \dot{y}$, and the Fluxion thereof will be $y\dot{x} + x\dot{y} + \dot{x}\dot{y}$, that is, $* y\dot{x} + x\dot{y}$: because $\dot{x}\dot{y}$ is a Quantity infinitely small, in respect of the other Terms $y\dot{x}$ and $x\dot{y}$: For if, for Example, you divide $y\dot{x}$ and $\dot{x}\dot{y}$ by $\dot{x}$, we shall have the Quotients y and $\dot{y}$, the latter of which is infinitely less than the former.

* Art. 2.

Whence it follows, *that the Fluxion of the Product of two Quantities, is equal to the Product of the Fluxion of the first of those Quantities in the second Plus the Product of the Fluxion of the second into the first.*

2. The Fluxion of xyz is $yz\dot{x} + xz\dot{y} + xy\dot{z}$. For by considering the Product xy as one Quantity, we must (as has been before shewn) take the Product of the Fluxion $y\dot{x} + x\dot{y}$ into the second Quantity z , (which will be $yz\dot{x} + xz\dot{y}$) Plus the Product of the Fluxion $\dot{z}$ of the second Quantity z into the first Quantity xy (which is $xy\dot{z}$;) and therefore the Fluxion of xyz will be $yz\dot{x} + xz\dot{y} + xy\dot{z}$.

3. The Fluxion of $xyz u$ is $u y z \dot{x} + u x z \dot{y} + u x y \dot{z} + x y z \dot{u}$. Which is proved as in the Case foregoing, by considering the Product xyz as one Quantity. Understand the same of others. Hence we have this

R U L E II.

For the Fluxions of Quantities of several Dimensions.

The Fluxion of a Quantity of several Dimensions, or (which is the same) of the Product of several Quantities multiply'd into one another, is equal to the Sum of the Products

B 3

of

of the Fluxion of each of those Quantities into the Product of the others.

So the Fluxion of ax is $x\dot{a} + a\dot{x}$, that is, $a\dot{x}$; and that of $a + x \times b - y$ is $b\dot{x} - y\dot{x} - a\dot{y} - x\dot{y}$.

PROP. III.

6. *TO find the Fluxion of a Fraction.*

The Fluxion of $\frac{x}{y}$ is $\frac{y\dot{x} - x\dot{y}}{yy}$. For sup-

posing $\frac{x}{y} = z$, then will x be $= yz$; and since these variable Quantities x and yz ought always to be equal to one another, whether they increase or decrease, it follows that their Fluxions, that is, their infinitely small Increments or Decrements shall likewise be equal.

Art. 5. And therefore will $\dot{x} = y\dot{z} + z\dot{y}$, and $\dot{z} = \frac{\dot{x} - z\dot{y}}{y} = \frac{y\dot{x} - x\dot{y}}{yy}$ by substituting z for its Value $\frac{x}{y}$. Which was to be done, &c. Hence we have this

RULE III.

For the Fluxions of Quantities divided by one another, or Fractions.

The Fluxion of a Fraction is equal to the Product of the Fluxion of the Numerator into the Denominator *Minus* the Product of the Fluxion of the Denominator into the Numerator: the whole being divided by the Square of the Denominator.

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As the Fluxion of $\frac{a}{x}$ is $-\frac{ax}{xx}$, and that of $\frac{x}{a+x}$ will be $\frac{ax}{aa+2ax+xx}$.

PROP. IV.

7. **T**O find the Fluxion of any whole or broken Powers of a variable Quantity.

Before we lay down a general Rule for finding the Fluxions of all sorts of Powers, we must explain the Analogy that there is between their Exponents or Indices. In order to this, if there be a Geometrical Progression, having 1 for its first Term, and the second Term be any Quantity x ; and if the Indices or Exponents be orderly set under them, it is plain that the said Exponents will form an Arithmetical Progression.

Prog. Geom. 1, x , xx , x^3 , x^4 , x^5 , x^6 , x^7 , &c.

Prog. Arithm. 0, 1, 2, 3, 4, 5, 6, 7, &c.

And if the Geometrical Progression be continued downwards from Unity, and the Arithmetical Progression downwards from 0, the Terms of this last shall be the Exponents of those to which they answer in the other;

as -1 is the Exponent of $\frac{1}{x}$, -2 of $\frac{1}{xx}$, &c.

Prog. Geom. x , 1, $\frac{1}{x}$, $\frac{1}{xx}$, $\frac{1}{x^3}$, $\frac{1}{x^4}$, &c.

Prog. Arith. 1, 0, -1 , -2 , -3 , -4 , &c.

But if some new Term be brought into the Geometrical Progression, we must find an answerable Arithmetical one for the Exponent of it.

As if the Term be $\sqrt{x}$, its Exponent will be $\frac{1}{2}$: $\sqrt[3]{x}$, will have $\frac{1}{3}$ for its Exponent: $\sqrt[5]{x^4}$, $\frac{4}{5}$:

$$\frac{1}{\sqrt{x^5}}, -\frac{3}{2} : \frac{1}{\sqrt[3]{x^5}}, -\frac{5}{3} : \frac{1}{\sqrt{x^{12}}}, -\frac{7}{2}, \text{ \&c.}$$

So that these Expressions $\sqrt{x}$ and $x^{\frac{1}{2}}$, $\sqrt[3]{x}$ and $x^{\frac{1}{3}}$, $\sqrt[5]{x^4}$ and $x^{\frac{4}{5}}$, $\frac{1}{\sqrt{x^5}}$ and $x^{-\frac{5}{2}}$, &c. signify the same thing.

Progr. Geom. 1, $\sqrt{x}$, x , 1, $\sqrt[3]{x}$, $\sqrt[3]{xx}$, x , 1, $\sqrt[5]{x}$, $\sqrt[5]{xx}$, $\sqrt[5]{x^3}$, $\sqrt[5]{x^4}$, x .

Progr. Arith. 0, $\frac{1}{2}$, 1, 0, $\frac{1}{3}$, $\frac{2}{3}$, 1, 0, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, 1.

Prog. Geom. $\frac{1}{x}$, $\frac{1}{\sqrt{x^3}}$, $\frac{1}{xx}$, $\frac{1}{x}$, $\frac{1}{\sqrt{x^4}}$, $\frac{1}{\sqrt[3]{x^5}}$, $\frac{1}{xx}$, $\frac{1}{x^3}$, $\frac{1}{\sqrt{x^7}}$, $\frac{1}{x^4}$.

Prog. Ar. -1 , $-\frac{1}{2}$, -2 , -1 , $-\frac{4}{3}$, $-\frac{5}{3}$, -2 , -3 , $-\frac{7}{2}$, -4 .

Where you may observe, that as $\sqrt{x}$ is a Geometrical Mean between 1 and x , $\frac{1}{2}$ is an Arithmetical Mean between their Exponents 0 and 1. In like manner, as $\sqrt[3]{x}$ is the first of two mean Proportionals between 1 and x , so $\frac{1}{3}$ is the first of two Arithmetical Means between their Exponents 0 and 1: Understand the same of others. Now from the Nature of these two Progressions it follows,

1. That the Sum of the Exponents of any two Terms of the Geometrical Progression, is the Exponent of the Product of these two Terms drawn into each other, as $x^4 + 3$ or x^7 is the Product of x^3 into x^4 , and $x^{\frac{1}{2}} + \frac{1}{2}$ or x^1 is the Product of $x^{\frac{1}{2}}$ into $x^{\frac{1}{2}}$, and $x^{-\frac{1}{2}} + \frac{1}{2}$ or $x^{-\frac{1}{2}}$ is the Product of $x^{-\frac{1}{2}}$ into $x^{\frac{1}{2}}$, &c. In like manner $x^{\frac{1}{2}} + \frac{1}{3}$ or $x^{\frac{5}{6}}$ is the Product of $x^{\frac{1}{2}}$ drawn into itself, that is, the Square of it,

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it, and $x^2 + x^2 + x^2$ or x^6 is the Product of x^2 into x^2 into x^2 , that is, the Cube of it, and $x^{-\frac{1}{3}} - \frac{1}{3} - \frac{1}{3} - \frac{1}{3} - \frac{1}{3}$, or $x^{-\frac{4}{3}}$, is the fourth Power of $x^{-\frac{1}{3}}$, and so of other Powers. Hence it is manifest, that the double, triple, &c. of the Exponent of any Term of the Geometrical Progression, is the Exponent of the Square, Cube, &c. of that Term; and consequently that the $\frac{1}{2}$, $\frac{1}{3}$, &c. of the Exponent of any Term of the Geometrical Progression, shall be the Exponent of the Square Root, Cube Root, &c. of that Term.

2. That the Difference of the Exponents of any two Terms of the Geometrical Progression, shall be the Exponent of the Quotient arising by the Division of one of those Terms by the other, as $x^{\frac{1}{2}} - \frac{1}{3} = x^{\frac{1}{6}}$ shall be the Exponent of the Quotient of $x^{\frac{1}{2}}$ divided by $x^{\frac{1}{3}}$, and $x^{-\frac{1}{3}} - \frac{1}{4} = x^{-\frac{7}{12}}$ shall be the Exponent of the Quotient of the Division of $x^{-\frac{1}{3}}$ divided by $x^{\frac{1}{4}}$; whence you see that it is the same thing to multiply $x^{-\frac{1}{3}}$ by $x^{-\frac{1}{4}}$, as to divide $x^{-\frac{1}{3}}$ by $x^{\frac{1}{4}}$: and so of others. This being well understood, there may happen two Cases in finding the Fluxions of Powers,

1. When the Power is a whole one, that is, when its Exponent is a whole Number. Now the Fluxion of xx is $2x\dot{x}$, of x^3 is $3xx\dot{x}$, of x^4 is $4x^3\dot{x}$, &c. for since the Square of x is only the Product of x into x , the Fluxion thereof shall be $* x\dot{x} + x\dot{x}$, that is, $2x\dot{x}$. In like manner, since the Cube of x is only the Product of x into x into x , the Fluxion of it will

* Art. 5.

* Art. 5. will be $x\dot{x}x + x\dot{x}x + x\dot{x}x$, that is, $3x\dot{x}x$; and since it is the same of any other of these Powers whatsoever, it follows, that if m represents any whole Number, the Fluxion of x^m will be $m x^{m-1} \dot{x}$.

If the Exponent be negative, the Fluxion of x^{-m} or of $\frac{1}{x^m}$ shall be $\frac{-m x^{m-1} \dot{x}}{x^{2m}} = -m x^{-m-1} \dot{x}$.

2. When the Power is broken, that is, when the Exponent is a Fraction. It is requir'd to find the Fluxion of $\sqrt[n]{x^m}$ or $x^{\frac{m}{n}}$ ($\frac{m}{n}$ be-

ing any Fraction.) Suppose $x^{\frac{m}{n}} = z$, and by raising both Sides of the Equation to the Power n , we shall have $x^m = z^n$, and finding the Fluxions of both Sides, (as in the first Case) then will $m x^{m-1} \dot{x} = n z^{n-1} \dot{z}$, and $\dot{z} = \frac{m x^{m-1} \dot{x}}{n z^{n-1}} = \frac{m}{n} x^{\frac{m}{n}-1} \dot{x}$, or $\frac{m}{n} \dot{x} \sqrt[n]{x^{m-n}}$, by

substituting $n x^{\frac{m}{n}-1}$ for its Value $n z^{n-1}$.

If the Exponent be negative, the Fluxion of

$$x^{-\frac{m}{n}} \text{ or of } \frac{1}{x^{\frac{m}{n}}} \text{ shall be } \frac{-\frac{m}{n} x^{\frac{m}{n}-1} \dot{x}}{x^{\frac{2m}{n}}} = -\frac{m}{n} x^{-\frac{m}{n}-1} \dot{x}.$$

From hence we have the following general

RULE IV.

For the Fluxions of all Kinds of Powers.

The Fluxion of any Power (whole or broken) of a variable Quantity, is equal to the Product

of FLUXIONS.

II

Product of the Exponent of that Power made by that same Quantity, raised to a Power less'n'd by 1, and multiply'd by its Fluxion.

As if m expresses any whole Number or Fraction positive or negative, and x any variable Quantity, the Fluxion of x^m will be always $m x^{m-1} \dot{x}$.

EXAMPLES.

The Fluxion of the Cube of $ay - xx$, that is, of $ay - xx^3$, is $3 \times ay - xx^2 \times a \dot{y} - 2xx \dot{x} = 3a^3 y y \dot{y} - 6a a x x y \dot{y} + 3a x^4 \dot{y} - 6a a y y x \dot{x} + 12a y x^3 \dot{x} - 6x^5 \dot{x}$.

The Fluxion of $\sqrt{xy + yy}$ or of $xy + yy^{\frac{1}{2}}$, is $\frac{1}{2} \times xy + yy - \frac{1}{2} \times y \dot{x} + x \dot{y} + 2y \dot{y}$, or $y \dot{x} + x \dot{y} + 2y \dot{y}$.

$$2\sqrt{xy + yy}$$

The Fluxion of $\sqrt{a^4 + axyy}$ or of $a^4 + axyy^{\frac{1}{2}}$, is $\frac{1}{2} \times a^4 + axyy - \frac{1}{2} \times ayy \dot{x} + 2ax y \dot{y}$, or $ayy \dot{x} + 2ax y \dot{y}$.

$$2\sqrt{a^4 + axyy}$$

The Fluxion of $\sqrt[3]{ax + xx}$ or of $ax + xx^{\frac{1}{3}}$, is $\frac{1}{3} \times ax + xx - \frac{1}{3} \times a \dot{x} + 2x \dot{x}$, or $a \dot{x} + 2x \dot{x}$.

$$3\sqrt[3]{ax + xx}$$

The Fluxion of $\sqrt{ax + xx} + \sqrt{a^4 + axyy}$ or of $ax + xx + \sqrt{a^4 + axyy}^{\frac{1}{2}}$, is $\frac{1}{2} \times ax + xx + \sqrt{a^4 + axyy} - \frac{1}{2} \times a \dot{x} + 2x \dot{x} + \frac{ayy \dot{x} + 2ax y \dot{y}}{2\sqrt{a^4 + axyy}}$, or

$$\frac{a \dot{x} + 2x \dot{x}}{2\sqrt{ax + xx} + \sqrt{a^4 + axyy}} + \frac{ayy \dot{x} + 2ax y \dot{y}}{2\sqrt{a^4 + axyy}}$$

$$2\sqrt{a^4 + axyy} \times 2\sqrt{ax + xx} + \sqrt{a^4 + axyy}$$

Lastly,

Lastly, the Fluxion of $\frac{\sqrt[3]{ax+xx}}{\sqrt{xy+yy}}$ accord-

• Art. 6, 7. ing to this Rule shall be *

$$\frac{\frac{ax+2xx}{3\sqrt[3]{ax+xx}} \times \sqrt{xy+yy} - \frac{yx+xy-2yy}{2\sqrt{xy+yy}} \times \sqrt[3]{ax+xx}}{xy+yy}.$$

SCHOLIUM.

8. **H**ERE we are to observe, that in finding the Fluxions of Quantities, we have hitherto supposed one of the variable Quantities x as increasing; while the others y , z , &c. do so likewise: That is, when the x 's become $x + \dot{x}$, the y 's and z 's, &c. become $y + \dot{y}$, $z + \dot{z}$, &c. So that if it should happen that some of them do decrease, while others increase, then the Fluxions of the former must be look'd upon as negative Quantities, with respect to the Fluxions of the latter increasing Quantities; and consequently the Signs of the Terms affected with the Fluxions of these decreasing Quantities, must be changed.

To make this plain, let x increase while y and z decrease; that is, when x becomes $x + \dot{x}$, let y and z become $y - \dot{y}$ and $z - \dot{z}$: First, get the Fluxion $x\dot{y}\dot{z} + xz\dot{y} + yz\dot{x}$ (by Art. 5.) and change the Signs of the Terms affected with $\dot{y}$ and $\dot{z}$. And this Expression thus alter'd, viz. $yz\dot{x} - x\dot{y}\dot{z} - xz\dot{y}$, shall be the Fluxion sought.

SECT.



S E C T. II.

*Of the Use of Fluxions in drawing
Tangents to all kinds of Curve Lines.*

D E F I N I T I O N.

IF one of the small Sides Mm of a Poly- *Fig. 2.*
gon, whereof a Curve Line consists *, be * *Art. 3.*
continued out; the said small Side thus conti-
nued, is a Tangent to the Curve in the Point
 M or m .

P R O P. I.

9. **I**T is required to draw a Tangent MT from *Fig. 3.*
a given Point M , in a Curve Line AM ,
whose Nature is expressed by any Equation, re-
presenting the relation of an Absciss AP to its
Correspondent ordinate PM .

Draw the Ordinate MP , and conceive the
right Line MT meeting the Diameter in the
Point T , to be the Tangent sought; more-
over, let mp be an Ordinate infinitely near
 MP , and draw the small right Line MR
parallel to AP . Now call the given Lines
 AP, x ; PM, y ; and then will Pp or MR
be $= \dot{x}$, and $Rm = \dot{y}$, and the similar Tri-
angles mRM, MPT , will give this Propor-
tion $mR(\dot{y}) : RM(\dot{x}) :: MP(y) : TP$
 $= \frac{y\dot{x}}{\dot{y}}$. Now by Means of the Fluxion of
the

the given Equation, we can get a Value of $\dot{x}$, expressed in Terms that will be all affected with $\dot{y}$; which being multiplied by y , and divided by $\dot{y}$, will give us the Value of the Subtangent PT in known Terms, freed from Fluxions; by which the sought Tangent MT may be drawn.

SCHOLIUM.

FIG. 4. 10. **W**HEN the Point T falls on the contrary Side of the Point A , where the x 's begin, with respect to P , then it is plain, that while x increases, y does decrease; and consequently, the Signs of all the Terms of the Fluxion of the given Equation affected with $\dot{y}$, must be * changed: Otherwise, the Value of $\dot{x}$ in $\dot{y}$ will be negative; and therefore that of $PT \left(\frac{y \dot{x}}{\dot{y}} \right)$. Notwithstanding, to avoid Trouble, it will be best to get the Fluxion of the given Equation, according to the Rules laid down in *Se&.* I. without any Alteration: For, if at the End of the Operation, it happens that the Value of PT be positive, then the Point T must be assumed on the same Side the Vertex A of the Diameter, as was supposed in the Operation; but if it be negative, it must be taken on the other Side the Point A . All this will be plain by the following Examples.

* Art. 8.

EXAMPLE I.

FIG. 3. II. 1. **I**F $ax = yy$ expresses the Relation of AP to PM , the Curve AM will be a Parabola, the given Quantity a , being the Parameter;

Parameter; and if you throw both Sides of the Equation into Fluxions, then will $a \dot{x} b \dot{y} = 2 y \dot{y}$, and $\dot{x} = \frac{2 y \dot{y}}{a}$, and $P T \left(\frac{y \dot{x}}{\dot{y}} \right) = \frac{2 y \dot{y}}{a} = 2 x$, by substituting $a x$ for $y \dot{y}$ the Value thereof. Hence, if you make $P T$ equal to the double of $A P$, and draw the right Line $M T$, this shall be a Tangent to the Curve in the Point M .

2. If $a a = x y$, be an Equation expressing FIG. 4 the Nature of an Hyperbola between the Asymptotes. By throwing both Sides of the Equation into Fluxions, we shall have $x \dot{y} + y \dot{x} = 0$; and therefore $P T \left(\frac{y \dot{x}}{\dot{y}} \right) = -x$. Whence, if you take $P T = P A$ on the other Side of the Point A , and draw the right Line $M T$, this will be a Tangent to the Hyperbola in the Point M .

3. Let $y^m = x$ express the Nature of all Kinds of Parabolas, where the Exponent m , represents a positive whole Number or Fraction. As also of all Kinds of Hyperbola's, when that represents a negative Number or Fraction. Now the Equation being thrown into Fluxions, will be $m y^{m-1} \dot{y} = \dot{x}$; and therefore $P T \left(\frac{y \dot{x}}{\dot{y}} \right) = m y^m = m x$, by substituting x for y^m , which is equal to it. If $m = \frac{2}{3}$, the Equation will be $y^{\frac{2}{3}} = x$, expressing the Nature of one of the Cubick Parabola's, and the Sub-tangent $P T = \frac{3}{2} x$. If $m = -2$, the Equation will be $a^2 = x y y$, expressing the Nature of a Cubick Hyperbola, and the Sub-tangent $P T = -2 x$.

If

If a Tangent be to be drawn to the Point A the Vertex of the Diameter of a Parabola, you must find the Ratio of $\dot{x}$ to $\dot{y}$ in that Point. For it is plain, when that Ratio is known, that the Angle made by the Tangent and the Diameter, will also be determined. In this Example, $\dot{x} : \dot{y} :: m y^{m-1} : 1$. Whence it appears, that y being $= 0$ in A , the Ratio of $\dot{y}$ to $\dot{x}$ will be then infinitely great, when m is greater than 1; and infinitely small when the same is less: That is, the Tangent to the Curve in A , must, in the former Case, be parallel to the Ordinates, and coincide with the Diameter in the latter Case.

EXAMPLE II.

FIG. 5. 12. IF AMB be a Curve Line of such a Nature, that $AP \times PB \left(x \times \overline{a-x} \right) :$

$\overline{PM}^2 (y y) :: AB (a) : AD (b)$. Then

$\frac{a y \dot{y}}{b} = a \dot{x} - x \dot{x}$; and throwing both Sides of the Equation into Fluxions, we have $\frac{2 a y \dot{y}}{b}$

$= a \dot{x} - 2 x \dot{x}$; whence $PT \left(\frac{y \dot{x}}{\dot{y}} \right) = \frac{2 a y \dot{y}}{a b - 2 b x}$
 $= \frac{2 a x - 2 x \dot{x}}{a - 2 x}$, by substituting $a x - x \dot{x}$ for

$\frac{a y \dot{y}}{b}$; and $PT - AP$, or $AT = \frac{a x}{a - 2 x}$.

Now if $\overline{AP}^3 \times \overline{PB}^2 \left(x^3 \times \overline{a-x}^2 \right) : \overline{PM}^5$

$(y^5) :: AB (a) : AD (b)$; then will $\frac{a y^5}{b}$ be
 $= x^3 \times \overline{a-x}^2$, and throwing both Sides of this
 Equation

Equation into Fluxions, we have $\frac{ay^4 \dot{y}}{b}$
 $= 3x\dot{x} \times a - x^2 - 2a\dot{x} + 2x\dot{x} \times x^3$; and so
 $\frac{y\dot{x}}{\dot{y}} = \frac{5x^3 \times a - x^2}{3xx \times a - x^2 - 2a + 2x \times x^3} =$
 $\frac{5x \times a - x}{3a - 3x - 2x}$, or $\frac{5ax - 5xx}{3a - 5x}$, and $AT =$
 $\frac{2ax}{3a - 5x}$.

AND generally, if m be the Exponent of the Power of AP , and n that of the Power of PB , we shall have $\frac{ay^{m+n}}{b} = x^m \times a - x^n$,

which is a general Equation for all Kinds of Ellipses. And throwing both Sides of this Equation into Fluxions, we have $\frac{m + n \times ay^{m+n-1} \dot{y}}{b}$

$= mx^{m-1} \dot{x} \times a - x^n - n \dot{x} \times x^{n-1}$, and so

by substituting $x^m \times a - x^n$ for $\frac{ay^{m+n}}{b}$, the Value thereof, there will come out $PT \left(\frac{y\dot{x}}{\dot{y}} \right)$

$= \frac{m + nx^m \times a - x^n}{mx^{m-1} \times a - x^n - n \dot{x} \times x^{n-1} \times x^m} =$
 $\frac{m + nx \times a - x}{ma - x - nx}$, or $PT = \frac{m + n \times ax - xx}{ma - m - nx}$,

and $AT = \frac{nax}{ma - m - nx}$.

EXAMPLE III.

13. **T**HE same Things being supposed as in FIG. 6. the foregoing Example, only here the Point B falls on the other Side A , with respect

spect to the Point P , and we shall have this Equation $\frac{a y^{m+n}}{b} = x^m \times a + x^n$, which expresses the Nature of all Hyperbolas with respect to their Diameters. Whence, as above,

$$\text{we get } PT = \frac{m+n \times ax + x^x}{ma + m + nx}, \text{ and } AT = \frac{nax}{ma + m + nx}.$$

Now if AP be supposed to be infinitely great, the Tangent TM will meet the Curve at an infinite Distance; that is, it will become the Asymptote CE ; in which Case AT $\left(\frac{nax}{ma + m + nx} \right)$ will be $= \frac{n}{m+n} a = AC$; because a being infinitely less than x , the Term ma will be o in regard to $m + nx$. By the same Reason, the Equation of the Curve here will be $a y^{m+n} = b x^{m+n}$. And making $m+n=p$, for Brevity sake, and extracting the Root p of both Sides, then will $y \sqrt[p]{a} = x \sqrt[p]{b}$, the Fluxion of which will be $y \sqrt[p]{a} = \sqrt[p]{x} \dot{b}$. So that if you draw AE parallel to the Ordinates, and conceive a small Triangle to be at the Point wherein the Asymptote CE meets the Curve, the following Proportion will arise $\dot{x} : \dot{y} :: \sqrt[p]{a} : \sqrt[p]{b} :: AC \left(\frac{n}{p} a \right) : AE = \frac{n}{p} \sqrt[p]{b} a^{p-1}$. Now the Value of CA and CE being thus determined, the Asymptote CE may be drawn.

If $m=1$, and $n=1$, the Curve will be the common Hyperbola, and AC will be $= \frac{1}{2} a$; and $AE = \frac{1}{2} \sqrt{ab}$; that is, one half of the conjugate Diameter, which is a known Truth established from other Principals.

EXAMPLE

EXAMPLE IV.

14. **L**ET there be an Equation $y^3 - x^3 = axy$, FIG. 6
(AP being $= x$, $PM = y$, and a is
a given right Line) expressing the Nature of
the Curve AM . The Fluxion of this shall be

$$3yy\dot{y} - 3xx\dot{x} = ax\dot{y} + ay\dot{x}. \text{ Whence } \frac{y\dot{x}}{\dot{y}} = \frac{3y^3 - axy}{3xx + ay}, \text{ and } AT \left(\frac{y\dot{x}}{\dot{y}} - x \right) =$$

$$\frac{3y^3 - 3x^3 - 2axy}{3xx + ay} = \frac{axy}{3xx + ay} \text{ by substituting } 3axy \text{ for } 3y^3 - 3x^3.$$

Now if AP and PM be supposed infinitely great, the Tangent TM will become the Asymptote CE , and the right Lines AT , AS , will become AC , AE , and these will determine the Position of the Asymptote. Now

$$AT, \text{ which call } t, \text{ is } = \frac{axy}{3xx + ay}, \text{ from whence}$$

$$\text{we get } y = \frac{3txx}{ax - at} = \frac{3tx}{a} \text{ when } AT \text{ becomes}$$

AC , because then at is 0 with respect to ax .

And putting down $\frac{3tx}{a}$ for y in $y^3 - x^3 =$

axy , and there will come out $27t^3x^3 - a^3x^3 = 3a^3txx$, and striking out the Term $3a^3txx$, since x being infinite, it is 0 with regard to the two others $27t^3x^3$ and a^3x^3 ; then will $AC(t)$ be $= \frac{1}{3}a$. In like manner AS

$$\left(y - \frac{x\dot{y}}{\dot{x}} \right) \text{ which call } s, \text{ is } = \frac{axy}{3yy - ax};$$

$$\text{whence we get } x = \frac{3syy}{ay + as} = \frac{3sy}{a}, \text{ since } y$$

being infinite with respect to s , the Term as will be 0 compar'd to ay ; and putting that

Value in the Equation of the Curve, we shall get $AE(s) = \frac{1}{3}a$. Whence if you take $AC = AE = \frac{1}{3}a$, and draw the right Line CE , it will be the Asymptote to the Curve AM .

These two latter Examples will serve as Guides in finding the Asymptotes of other Curve Lines.

P R O P. II.

FIG. 7. 15. **I**F in the Proposition foregoing, the Abscisses AP be conceived to be Parts of a Curve Line that we know how to draw the Tangents (PT) of. It is requir'd to draw the Tangent MT from the given Point M in the Curve AM .

Draw the Ordinate MP , and the Tangent PT , and suppose the right Line MT , which cuts it in T , to be the Tangent sought: likewise suppose another Ordinate mp infinitely near to the former, and let the little right Line MR be parallel to PT : then call the given Quantities AP, x , and PM, y , and as before we shall have Pp or $MR = \dot{x}$, $Rm = \dot{y}$, and because of the similar Triangles mRM, MPT , we have this Proportion, viz. $mR(\dot{y}) : RM(\dot{x}) :: MP(y) : PT = \frac{y\dot{x}}{\dot{y}}$. Now proceed with the Equation expressing the Relation of the Abscisses $AP(x)$ to the Ordinates $PM(y)$, as in the foregoing Examples, and moreover the following ones.

E X A M P L E I.

16. **L**ET $\frac{yy}{x}$ be $= \frac{x\sqrt{aa+yy}}{a}$: this thrown into

into Fluxions will be $\frac{2xy\dot{y} - yy\dot{x}}{xx} = \frac{\dot{x}\sqrt{aa+yy}}{a}$

$+\frac{xy\dot{y}}{a\sqrt{aa+yy}}$, and reducing the same into an

Analogy $\dot{y} : \dot{x} (MP : PT) :: \frac{\sqrt{aa+yy}}{a}$

$+\frac{yy}{xx} : \frac{2xy}{xx} - \frac{xy}{a\sqrt{aa+yy}}$. And therefore the

Relation of the given Ordinate MP to the Subtangent PT sought, is expressed in known Terms freed from Fluxions. Which is what was proposed to be done.

EXAMPLE II.

17. **L**ET x be $=\frac{ay}{b}$. This thrown into

Fluxions, and we have $\dot{x} = \frac{a\dot{y}}{b}$; and PT

$\left(\frac{y\dot{x}}{\dot{y}}\right) = \frac{ay}{b} = x$. If the Curve APB be a Se-

micircle, and the Ordinates MP , continued out to Q , be perpendicular to the Diameter AB ;

then the Curve AMC shall be a Semi-Cycloid. And when $b=a$, the Cycloid will be a com-

mon one: when b is greater than a , the Cy-

cloid will be a *Prolate* one, and when it is less, a *Curtate* one.

COROL.

18. **W**HEN the generating Point of the Cycloid is in the Periphery of the Circle, if you draw the Chord AP . I say, it will be parallel to the Tangent MT . For the Triangle MPT being then an Isosceles one,

C 3

the

the external Angle TPQ shall be the Double of the internal opposite Angle TMQ . But the Angle APQ is equal to the Angle APT , because half of the Arch AP is the Measure of each of them; and therefore it is the one half of the Angle TPQ . Whence the Angles TMQ , APQ , shall be equal to each other; and consequently the Lines MT , AP shall be parallel.

P R O P. III.

FIG. 7. 19. **L**ET AP be any Curve Line, and the right Line $KNAQ$ a Diameter of it; and supposing the Method of drawing Tangents (PK) to it known; likewise let AM be another Curve such, that drawing any how the Ordinate MQ cutting the former Curve in the Point R , the relation of the Arch AP to the Ordinate MQ be expressed by any Equation. It is requir'd to draw the Tangent MN from a given Point M .

Call the known Quantities PK, t ; KQ, s ; the Arch AP, x ; MQ, y : then supposing another Ordinate mq infinitely near MQ , and drawing PO, MS parallel to AQ , we shall have $Pp = x$, $mS = y$, and since the Triangles KPQ and Ppo , mSM and MQN are similar; therefore $PK(t) : KQ(s) :: Pp(x) : PO$ or $MS = \frac{sx}{t}$. And $mS(y) : SM\left(\frac{sx}{t}\right) :: MQ$

$(y) : QN = \frac{syx}{ty}$. Now by means of the Fluxion of the given Equation find a Value of x in Terms that are all affected by y , and if you substitute this Value in $\frac{syx}{ty}$ for x , the y^s will

vanish,

vanish, and the Quantity of the Subtangent QN sought, will be expressed in known Terms. Which was to be found.

PROP. IV.

20. **L**ET there be two Curves AQC, BCN , FIG. 8. the right Line $TEABF$ being a Diameter; and suppose the Method of drawing the Tangents QE, NF , to be known; moreover, let there be another Curve Line MC such, that the Relation of the Ordinates MP, QP, NP , be expressed by any given Equation. It is required to draw the Tangent MT from a given Point M in this latter Curve.

IMAGINE the small Triangles QOq, MRm, NSn , at the Points Q, M, N , and call the known Quantities $PE, s; PF, t; PQ, x; PM, y; PN, z$; then will Oq be $= \dot{x}$, $Rm = \dot{y}$, $Sn = -\dot{z}$, because * when * Art. 8. x and y increase, z decreases. And since the Triangles QPE and qOQ , NPF and nSN , MPT and mRM are similar; therefore $QP(x) : PE(s) :: qO(\dot{x}) : OQ$ or MP or $SN = \frac{s\dot{x}}{x}$. And $NP(z) : PF(t) :: nS(-\dot{z}) : SN$

$$= \frac{-t\dot{z}}{z} = \frac{s\dot{x}}{x}. \left(\text{Whence arises } \dot{z} = \frac{-sz\dot{x}}{tx} \right)$$

And $mR(\dot{y}) : RM\left(\frac{s\dot{x}}{x}\right) :: MP(y) : PT = \frac{sy\dot{x}}{xy}$. Now if $-\frac{sz\dot{x}}{tx}$ be substituted for

in the Equation of the Curve thrown into Fluxions, we shall have a Value of $\dot{x}$ in y ;

C 4 which

which being put in $\frac{syx}{xy}$, and the y 's will destroy one another; and so the Value of the Subtangent PT will be had in known Terms.

EXAMPLE.

21. **L**ET $yy = xz$; this thrown into Fluxions, is $2y\dot{y} = z\dot{x} + x\dot{z} = \frac{tzx - szx}{t}$

by putting $-\frac{szx}{tz}$ for $\dot{z}$; whence we get $\dot{x} = \frac{2ty\dot{y}}{tz - sz}$; and therefore $PT\left(\frac{syx}{xy}\right) =$

$$\frac{2sty y}{txz - szx} = \frac{2st}{t - s}, \text{ by substituting } xz \text{ for } yy.$$

Again, let there be given this general Equation, viz. $y^{m+n} = x^m z^n$; this thrown into Fluxions will be $m + ny^{m+n-1}\dot{y} = m z^n x^{m-1}\dot{x} + n x^m z^{n-1}\dot{z} = \frac{mtz^n x^{m-1}\dot{x}}{t} - \frac{nsz^n x^{m-1}\dot{x}}{t}$,

by putting $-\frac{szx}{tx}$ for $\dot{z}$; whence we get $PT\left(\frac{syx}{xy}\right) = \frac{mst + nsty^{m+n}}{mtz^n x^m - nsz^n x^m} = \frac{mst + nst}{mt - ns}$, if $x^m z^n$ be put for y^{m+n} .

Here you may observe, that if the Curves AQC , BCN , become right Lines, the Curve MC will be one of the Conick Section kind, viz. an *Ellipsis*, when the Ordinate CD , drawn from the Point of Concurrence C , falls between the Extremities A and B ; an *Hyperbola*, when it falls on either Side; and a *Parabola*, when one of the Extremities A or B is infinitely distant from the other, or when one of the

the right Lines CA or CB is parallel to the Diameter AB .

PROP. V.

22. **L**ET APB be a Curve beginning at the Point A , and suppose the Method of drawing Tangents (PH) to it known; and let the Point F be assumed without this Curve, always having the same Situation, and if there be another Curve CMD such, that any right Line FMP being drawn, the Relation of the Part FM thereof, to the Part AP of the Curve, is expressed by any given Equation. It is required to draw the Tangent MT from the given Point M . FIG. 9.

Draw FH perpendicular to FP , meeting the given Tangent PH in the Point H , and the sought Tangent MT in the Point T , suppose a right Line $FrmOp$ making an Angle infinitely small with FP , and from the Centre F describe the small Arches PO, MR , of a Circle; the little Triangle pOp shall be similar to the right-angled Triangle PFH ; for the Angles HPF, HpF , are *equal, because * Art. 2. they differ only by the Angle Pfp , which is supposed to be infinitely small; and moreover the Angle pOp is a right Angle, since the Tangent in O (which is the Continuation of the little Arch PO consider'd as a right Line) is perpendicular to the Radius FO . By the same Reason, the Triangles mRM, MFT , will be similar. Now it is evident, that the little Triangles or Sectors FPO, FMR are similar. And call the known Quantities PH, t ; HF, s ; FM, y ; FP, z ; and the Arch AP, x ; then

then shall $PH(t) : HF(s) :: Pp(\dot{x}) : PO = \frac{s \dot{x}}{t}$. And $FP(z) : FM(y) :: PO \left(\frac{s \dot{x}}{t} \right) :$

$MR = \frac{ys \dot{x}}{tz}$. And $mR(\dot{y}) : RM \left(\frac{sy \dot{x}}{tz} \right) ::$

$FM(y) : FT = \frac{sy y \dot{x}}{tz \dot{y}}$. And by throwing the given Equation into Fluxions, what is still to be done may be effected.

EXAMPLE.

FIG. 10. 23. **I**F the Curve APB be a Circle, the Point F being the Centre; it is plain that the Tangent PH does become parallel and equal to the Subtangent FH , because HP shall be also perpendicular to PF ; and so, in this Case, $FT = \frac{yy \dot{x}}{z \dot{y}} = \frac{yy \dot{x}}{a \dot{y}}$, by calling $FP(z)$, a ; since it is now a constant Quantity. This being supposed, if the whole Periphery, or any determinate Part thereof, be called P , and you make $b : x :: a : y$, the Curve CMD , which in this Case is FMD , will be the Spiral of *Archimedes*, and $y = \frac{ax}{b}$, which thrown in Fluxions, will be $\dot{y} = \frac{a \dot{x}}{b}$, whence arises $y \dot{x} = \frac{by \dot{y}}{a} = x \dot{y}$, by putting $\frac{ax}{b}$ for y ; and therefore $FT \left(\frac{yy \dot{x}}{a \dot{y}} \right) = \frac{xy}{a}$. And so we get the following Construction.

From

From the Centre F with the Radius FM , describe the circular Arch $M\mathcal{Q}$, bounded in $\mathcal{Q}$ by the Radius FA joining the Points A, F ; and take $F\mathcal{Q} = \text{Arch } M\mathcal{Q}$. I say, the right Line MT will be a Tangent in M . For because of the similar Sectors $FP A, FM\mathcal{Q}$, the following Proportion will arise $FP (a) : FM (y) :: AP (x) : M\mathcal{Q} = \frac{y x}{a} = FT$.

If you suppose generally, that $b : x :: a^m : y^m$ (the Exponent m expressing any whole Number or Fraction) the Curve FMD will be a Spiral of all Kinds *ad infinitum*. And then $y^m = \frac{a^m x}{b}$, and this thrown into Fluxions, and

$$m y^{m-1} \dot{y} = \frac{a^m \dot{x}}{b}; \text{ whence arises } y \dot{x} = \frac{m b y^m \dot{y}}{a^m} \\ = m x \dot{y}, \text{ by putting } \frac{a^m x}{b} \text{ for } y^m; \text{ and there-} \\ \text{fore } FT \left(\frac{y y \dot{x}}{a \dot{y}} \right) = \frac{m x y}{a} = m \times M\mathcal{Q}.$$

PROP. VI.

24. **L**ET there be a Curve Line APB , the FIG. III.
Method of drawing Tangents (PH) to it being known, and let F be a given Point taken without the same, always keeping its Situation. Let there be likewise another Curve CMD such, that any how drawing the right Line FPM , the Relation of FP to FM be expressed by any given Equation. It is required to draw a Tangent MT from the given Point M .

Draw the right Line FHT perpendicular to FM , and suppose (as in the last Proposition)

tion) the little Triangles POp , MRm similar to the Triangles HFP , TFM ; then calling the known Quantities FH, s ; FP, x ; FM, y ; and we shall have $PF(x):FH(s)::pO(\dot{x})::$

$$OP = \frac{s\dot{x}}{x}. \text{ And } FP(x):FM(y)::OP\left(\frac{s\dot{x}}{x}\right)$$

$$:RM = \frac{sy\dot{x}}{xx}. \text{ And } mR(y):RM\left(\frac{sy\dot{x}}{xx}\right)::$$

$$FM(y):FT = \frac{sy\dot{y}}{xx\dot{y}}. \text{ What is farther to be}$$

done, may be effected by throwing the given Equation into Fluxions.

EXAMPLE.

25. **I**F instead of the Curve APB , you would have a straight Line PH , and the Equation expressing the Relation of FP to PM be $y - x = a$; that is, if PM be always equal to the same given right Line a ; then will $\dot{y}$ be $= \dot{x}$; and therefore $FT\left(\frac{sy\dot{y}}{xx\dot{y}}\right) = \frac{sy\dot{y}}{xx}$. Whence we get the following Construction.

Draw ME parallel to PH , and MT parallel to PE : I say, the same will touch the Curve in M .

$$\text{For } FP(x):FH(s)::FM(y):FE = \frac{s\dot{y}}{x}.$$

$$\text{And } FP(x):FE\left(\frac{s\dot{y}}{x}\right)::FM(y):FT = \frac{sy\dot{y}}{xx}.$$

It is therefore manifest, that the Curve GMD is the Conchoid of *Nicomedes*, the right Line PH being the Asymptote, and the Point F the Pole.

PROP.

PROP. VII.

26. **L**ET ARM be a Curve, the Method of FIG. 12. drawing Tangents (MH) to it being known, and let the right Line EPAHT be a Diameter thereof, without which let the Point F have a constant Position; and from the same let the indefinite right Line FPSM issue, cutting the Diameter in P, and the Curve in M. Now if you conceive the right Line FPM to revolve about the Point F, and at the same time to move the Plane PAM constantly parallel to itself along the indefinite immoveable right Line ET, so that the Distance PA be every where the same; it is evident that (M) the continual Intersection of the Lines FM, AM, by this Motion, will describe the Curve CMD. From the given Point M of which, it is required to draw MT to touch the Curve.

The Plane PAM being supposed to be come to a Situation *pam* infinitely near, and the right Line *mRS* drawn parallel to *AP*, it is evident (from the manner of Generation) that $Pp = Aa = Rm$; and therefore that $RS = Sm - Pp$. Now if you call the known Quantities *FP* or *Fp*, *x*; *FM* or *Fm*, *y*; *PH*, *s*; *MH*, *t*; and the Fluxion *Pp*, $\dot{z}$: Then from the similar Triangles *FPp* and *FSm*, *MPH* and *MSR*, *MHT* and *MRm*, we shall have

$$Fp(x) : Fm(y) :: Pp(\dot{z}) : Sm = \frac{y\dot{z}}{x} \left(\text{whence } SR = \frac{y\dot{z} - x\dot{z}}{x} \right). \text{ And } PH(s) : HM(t) :: SR \left(\frac{y\dot{z} - x\dot{z}}{x} \right) : RM = \frac{tyz - tx\dot{z}}{sx}. \text{ And } MR$$

$$\left(\frac{tyz - txz}{sx}\right) : Rm(z) :: MH(t) : HT = \frac{sx}{y-x}.$$

Whence if FE be drawn parallel to MH , and you take $HT = PE$; the Line MT shall be the Tangent sought.

If AM were a right Line, the Curve CMD would be an Hyperbola, and the Line ET would be one of the Asymptotes. And if it were a Circle, the Point P would be the Centre, and the Curve CMD would be the Conchoid of *Nicomedes*, the Line ET being its Asymptote, and the Point F the Pole of it. But if it were a Parabola, the Curve CMD would be one of the parabolick Kind, mentioned by *Descartes* in *Lib. 3. Geom.* and at the same time would be described below the right Line ET , by the Interfection of FM with the other half of the Parabola.

PROP. VIII.

FIG. 13. 27. **L**ET AN be a Curve, whose Diameter is AP , and let F be a Point without them, having a constant Situation: Moreover let CMD be another Curve such, that any how drawing the right Line $FMPN$, the Relation of the Parts FN , FP , FM of it, is expressed by any given Equation. It is required to draw a Tangent MT from any given Point M in it.

Thro' the Point F draw the Line HK perpendicular to FN , meeting the Diameter AP in K , and the given Tangent NH in H . From the Centre F , with the Distances FN , FP , FM , describe the small Arches NQ , PO , MR , terminated by the right Line Fn supposed to make an infinitely small Angle with FN . This being supposed.

I

Call

of FLUXIONS.

31

Call the known Quantities $FK, s; FH, t; FP, x; FM, y; FN, z$; then because of the similar Triangles PFK and pOP , FMR and FPO , FPO and FNQ , HFN and NQn , mRM and MFT , we shall have the following Proportions: $PF(x):FK(s)::pO(\dot{x}):OP=$

$$\frac{s\dot{x}}{x}. \text{ And } FP(x):FM(y)::PO\left(\frac{s\dot{x}}{x}\right):MR=$$

$$\frac{sy\dot{x}}{xx}. \text{ And } FP(x):FN(z)::PO\left(\frac{s\dot{x}}{x}\right):NQ$$

$$= \frac{sz\dot{x}}{xx}. \text{ And } HF(t):FN(z)::NQ\left(\frac{sz\dot{x}}{xx}\right)$$

$$:Qn(-\dot{x}) = \frac{sz\dot{x}}{txx}. \text{ And } mR(y):RM\left(\frac{sy\dot{x}}{xx}\right)$$

$$::FM(y):FT = \frac{sy\dot{x}}{xx\dot{y}}. \text{ Now by throwing}$$

the given Equation into Fluxions, we shall find a Value of $\dot{y}$ in $\dot{x}$ and $\dot{z}$; in which substituting

$$\frac{-sz\dot{x}}{txx} \text{ for } \dot{z}; \text{ because when } x \text{ increases, } z \text{ decreases; and then all the Terms will be affected with } \dot{x}: \text{ So that at length this Value being put in } \frac{sy\dot{x}}{xx\dot{y}}, \text{ the } \dot{x}\text{'s will go out; and therefore the Value of } FT \text{ will be expressed in known Terms freed from Fluxions.}$$

If the right Line AP were supposed a Curve, and the Tangent PK had been drawn, we should find that FT would always have the same Value, and the Reasoning would have been the same.

EXAMPLE

EXAMPLE.

FIG. 14. 28. **L**ET the Curve AN be a Circle passing thro' the Point F (so situated with respect to the Diameter AP , that the Line FB perpendicular to the same Diameter, passes thro' G the Centre of the Circle) and let PM be always equal to PN ; it is manifest that the Curve CMD , which in this Case becomes FMA , will be the Cissoïd of *Diocles*, and the Equation thereof $z+y=2x$; which thrown into Fluxions, will be $\dot{y}=2\dot{x}-\dot{z}=\frac{2txx\dot{x}+szz\dot{x}}{1xx}$, by substituting $-\frac{szz\dot{x}}{1xx}$ found above (*Art.* 27.) for $\dot{z}$. And therefore FT

$$\left(\frac{sy y \dot{x}}{xx y}\right) = \frac{syy}{21xx + szz}$$

If the Point M coincides with the Point A , the Lines FM , FN , FP , will be each equal to FA ; as also the right Lines FK , FH ; and therefore we shall have in this Case $FT = \frac{x^4}{3x^3} = \frac{1}{3}x$; that is, take $FT = \frac{1}{3}AF$, and draw the right Line AT , the same will touch the Curve in the Point A .

Tangents may be drawn likewise to the Cissoïd by means of the first Proposition, by drawing NE , ML , perpendicular to the Diameter FB , and seeking an Equation expressing the Relation between the Ordinate LM , and correspondent Absciss FL . And this may be thus done; first call the known Quantities FB , $2a$; FL or BE , x ; LM , y ; then from the Similarity of the Triangles FEN , FLM , and the Nature of the Circle, we have $FL(x) : LM(y) :: FE : FN :: EN(\sqrt{2ax - xx}) : EB(x)$.

EB(x). Whence we get $yy = \frac{x^3}{2a-x}$; which thrown into Fluxions, will be $2y\dot{y} = \frac{6ax\dot{x} - 2x^3\dot{x}}{2a-x^2}$; and therefore $LO^* \left(\frac{yx}{\dot{y}} \right) =$ * *Art. 9.*
 $\frac{yy \times 2a - x}{3axx - x^3} = \frac{2ax - xx}{3a - x}$, by putting $\frac{x^3}{2a-x}$ for yy .

PROP. IX.

29. **L**ET there be two Curves *ANB*, *CPD*, FIG. 15.
 and a straight Line *FKT*, in which are taken the three Points *A*, *C*, *F*; and let there be some other Curve *EMG* such, that a right Line *FMN* being drawn from any Point *M* of it, and the right Line *MP* parallel to *FK* the Relation of the Arch *AN* to the Arch *CP*, is expressed by any given Equation. It is required to draw the Tangent *MT* from a given Point *M* in the Curve *EG*.

Thro' the Point *T* sought, draw the right Line *TH* parallel to *FM*; and thro' the given Point *M*, the right Lines *MRK*, *MOH*, parallel to the Tangents in *P* and *N*, and draw *FmOn* infinitely near to *FMN*, and *mRp* parallel to *MP*.

Now call the known Quantities *FM*, s ; *FN*, t ; *MK*, u ; *CP*, x ; *AN*, y ; (then will Pp or $MR = \dot{x}$, $Nn = \dot{y}$). And because of the similar Triangles *FNn*, and *FMO*; *MOm*, and *MHT*; *MRm*, and *MKT*; therefore $FN(t) : FM(s) :: Nn(\dot{y}) : MO = \frac{s\dot{y}}{t}$.

And $MR(\dot{x}) : MO\left(\frac{s\dot{y}}{t}\right) :: MK(u) : MH = \frac{suy}{tx}$.

D

Now

Now by help of the Fluxion of the given Equation, we shall have a Value of $\dot{y}$ in Terms, every of which will be affected with $\dot{x}$; which being substituted in $\frac{su\dot{y}}{t\dot{x}}$, and the $\dot{x}$'s do destroy

one another. And therefore the Quantity of MH will be expressed in known Terms. From whence we have the following Construction.

Draw MH parallel to the Tangent in N , and equal to the Expression before found; draw HT parallel to FM , meeting the right Line FK in T ; through which, and the given Point M , you must draw the Tangent MT ; which will be that sought.

EXAMPLE.

FIG. 16. 30. IF the Curve ANB be a Quadrant of a Circle, the Point F being the Centre; and the Curve CPD becomes the Radius APF , perpendicular to the right Line $FKG \perp IB$, and the Arch $AN(y)$, be always to the right Line $AP(x)$, as the Quadrant $ANB(b)$, to the Radius $AF(a)$; then the Curve EMG , shall become AMG the *Quadratrix* of *Dionysius*; and $MH\left(\frac{su\dot{y}}{t\dot{x}}\right)$ will be $= \frac{as\dot{y} - sx\dot{y}}{a\dot{x}}$, since FP or $MK(u) = a - x$, and $FN(t) = a$. But from the supposed Analogy $ay = bx$, and $a\dot{y} = b\dot{x}$. Now putting $\frac{ay}{b}$ and $\frac{b\dot{x}}{a}$ for their Equals x and $\dot{y}$, in the Value of MH , and there arises $MH = \frac{bs - ys}{a}$. From whence we get the following Construction.

Draw

Draw MH perpendicular to FM , and equal to the Arch MQ described from the Centre F , and draw HT parallel to FM . I say, the Line MT will touch the Curve in M : For because of the similar Sectors FNB , FMQ , we have this Proportion, viz. $FN(a) : FM(s) :: NB(b-y) : MQ = \frac{bs-sy}{a}$.

COROLLARY.

31. IF you desire the Determination of the Point G , where the *Quadratrix* AMG meets the Radius FB , you must conceive another Radius Fgb infinitely near FGB ; and then drawing gf parallel to FB , from the Nature of the *Quadratrix*, and the similar Triangles FBb , gfF , right-angled at B and f , we shall have this Proportion $AB : AF :: Bb : Ff :: FB$ or $AF : gf$ or FG . FIG. 17.

Whence if you take a third Proportional to the Quadrant AB , and the Radius AF , it shall be equal to FG ; that is, $FG = \frac{aa}{b}$. By which Means the Construction of Tangents may be shortened thus:

Draw TE parallel to MH ; then from the similar Triangles FMK , FTE , we have MK FIG. 16.

$$(a-x) : MF(s) :: ET \text{ or } MH \left(\frac{bs-sy}{a} \right) : FT = \frac{bss-yss}{aa-ax} = \frac{bss}{aa}, \text{ by putting } \frac{ay}{b} \text{ for its}$$

Equal x , and afterwards dividing the whole by $b-y$. From whence it is manifest, that the Line FT is a third Proportional to FG and FM .

PROP. X.

FIG. 18. 32. **L**ET AMB be a Curve such, that the Relation of the right Lines $MF, MG, MH, \&c.$ drawn from any Point M taken in it, to the Foci $F, G, H, \&c.$ is expressed by any given Equation. It is required to draw the Perpendicular MP from the given Point M , to the Tangent in that Point.

Take the infinitely small Arch Mm in the Curve AB , and draw the right Lines FRm, GmS, HmO ; and from the Centres F, G, H , describe the small Arches MR, MS, MO ; likewise from the Centre M , with any Distance, describe the Circle CDE cutting the Lines MF, MG, MH , in the Points C, D, E ; from which let fall the Perpendiculars CL, DK, EI , to MP . This being done, you may observe,

1°, That the right-angled Triangles MRm, MLC , are similar; for if from the right Angles LMm, RMC , the common Angle LMR be taken away, the Angles remaining RMm, LMC are equal; and they are right-angled at R and L . After the same way we prove that the right-angled Triangles MSm and MKD , MOm and MIE , are similar. Therefore since Mm is a common Hypotenuse to the little Triangles MRm, MSm, MOm , and the Hypotenuses MC, MD, ME , of the Triangles MLC, MKD, MIE , are equal; it is evident that the Perpendiculars CL, DK, EI , have the same Relation to each other, as the Fluxions Rm, Sm, Om .

2°, That

2°, That the Lines issuing from the Foci situate on the same Side the Perpendicular MP , do increase at the same time the others decrease, or contrariwise. As (*in Fig. 18.*) FM increases by its Fluxion Rm , while GM, HM , decrease by theirs, *viz.* Sm, Om .

Now if the Equation $ax + xy - zz = 0$, be supposed to express, for Examples sake, the Relation of the right Lines $FM(x)$, $GM(y)$, $HM(z)$. This Equation thrown into Fluxions, will be $a\dot{x} + y\dot{x} + x\dot{y} - 2z\dot{z} = 0$. And then it will follow that the Tangent in M (which indeed is only the Continuation of the little Side Mm of the Polygon, that the Curve AMB is conceiv'd * to be made of) must be * *Art. 3.* so situate, that if the Parallels mR, mS, mO , to the right Lines FM, GM, HM , be drawn from any Point m in it, terminated in R, S , and O , by Perpendiculars MR, MS, MO , to the said right Lines, we shall always have this Equation $a + y \times Rm + x \times Sm - 2z \times Om = 0$; or (which comes to the same, in putting CL, DK, EI , for their Proportionals Rm, Sm, Om) the Perpendicular (MP) to the Curve, must be situate so, that $a + y \times CL + x \times DK - 2z \times EI = 0$. From whence we have the following Construction.

If the Point C be conceived to be laden with a Weight $a + y$, which multiples the Fluxion $\dot{x}$ of the right Line FM on which it is situate, and the Point D with the Weight x , and the Point E , taken on the contrary Side M with regard to the Focus H (since the Term $-2z\dot{z}$ is negative) with the Weight $2z$. I say, the right Line MP passing thro' the common Centre of Gravity of the Weights supposed to

FIG. 18,
19.

be in the Points C, D, E , will be the Perpendicular required.

FIG. 19.

For it is plain (by the Principles of Mechanics) that every right Line passing thro' the Centre of Gravity of several Weights, so separates them, that the Weights on one Side, each multiplied by its Distance from that Line, are equal to the Weights on the other Side, each multiplied also by its Distance from the said Line. Whence, supposing x, y, z to increase together; that is, conceiving the Foci F, G, H , to be all on one side MP , as we have always done, in throwing the given Equation into Fluxions, according to the Rules before delivered: Therefore the Line MP will leave on one Side the Weights in C and D , and on the other the Weight in E , and so we shall have $\overline{a+y} \times CL + x \times DK - 2z \times EI = 0$, which is the Equation to be constructed.

FIG. 18.

* Art. 8.

Now since the Construction is just in this Case, I say it is likewise so in any other Case. For Example, if the Situation of the Point M in the Curve be so alter'd, that x increases while y and z decrease; that is, that the Foci G and H fall on the other Side MP . Then it follows, 1°. * That the Signs of the Terms of the given Equation affected with y and z , or their Proportionals DK, EI , must be changed. Whence the Equation to be constructed in this latter Case, will be $\overline{a+y} \times CL - x \times DK + 2z \times EI = 0$. 2°. That the Weights, in D and E will change their Situation with regard to MP ; and so from the Nature of the Centre of Gravity, we have $\overline{a+y} \times CL - x \times DK + 2z \times EI = 0$, which is the Equation to be con-

constructed. And since this is so in all the Cases possible, therefore, &c.

Hence it appears, that the Reasoning is the same, let what will be the Number of the Foci, and the given Equation, so that we thus denounce the general Construction.

Find the Fluxion of one Side of the given Equation (the other Side being 0) and from the Centre M , with any convenient Distance, describe the Circle CDE intersecting the right Lines MF , MG , MH , in the Points C , D , E , in which are conceived the Weights that have the same Relation as the Quantities, multiplying the Fluxions of the Lines on which they are situate. I say, the Line MP passing through their common Centre of Gravity, shall be the Perpendicular required. But here we must observe, that if one of the Weights be negative in the Fluxion of the given Equation, the same must be supposed to be on the contrary Side of the Point M with regard to the Focus.

If instead of the Foci F , G , H , you suppose right Lines or Curves, to which the right Lines MF , MG , MH , fall at right Angles, the same Construction still takes place.

FIG. 20.

For drawing from the Point m taken infinitely near M the Perpendiculars mf , mg , mh , to the Foci, viz. the Lines in the Foci; and from the Point M , the little Perpendiculars MR , MS , MO , to those Lines. It is evident that Rm will be the Fluxion of MF , because the right Lines MF , Rf , being Perpendiculars between the Parallels Ff , MR , are equal to one another; and in like manner Sm is the Fluxion of MG , and Om that of MH ; and what remains may be proved as above.

FIG. 21.

Instead of all or some of the Foci F, G, H , we may suppose Curves beginning in F, G, H . And the Curve AMB such, that the Tangents MV, MX , and the right Line MG being drawn from any Point M in it; the Relation of the mixed Lines FVM, HXM , and the right Line GM be expressed by a given Equation. For by drawing the Tangent mu from the Point m taken infinitely near M , it is manifest that the same will meet the other Tangent in the Point V , (it being only the Continuation of the little Arch Vu , consider'd as a little right Line.) And therefore if from the Centre V be described the small Circular Arch MR ; Rm shall be the Fluxion of the mixed Line FVM , which becomes $FVuRm$. And all the rest may be demonstrated as foregoing.

This Problem was first started by Mr. Tschirnhausen, in his Book de la Medecine de l'esprit; and Mr. Fatio in the Journal des Scavans, has given a very ingenious Solution thereof. But their Solutions are only particular Cases of the general Construction here laid down.

EXAMPLE I.

FIG. 22.

33. **L**ET $axx + byy + czx - f = 0$, (the right Lines a, b, c, f being given) the Fluxion of which is $ax\dot{x} + by\dot{y} + cz\dot{z} = 0$. Now supposing the Weight ax in C , the Weight by in D , and the Weight cz in E , viz. Weights that are to one another, as those Rectangles. The Line MP going through the common Centre of Gravity of them, shall be perpendicular to the Curve in the Point M .

But

But if FO be drawn parallel to CL , and the Radius MC be $= 1$, then (because of the similar Triangles, MCL , MFO) FO will be $= x \times CL$; and in like manner drawing GR parallel to DK , and HS to EI , we shall have $GR = y \times DK$, and $HS = z \times EI$: so that by imagining the Weights a, b, c in the Foci F, G, H , the Line MP passing through the Centre of Gravity of the Weights ax, by, cz conceiv'd in C, D, E , shall also pass through the Centre of Gravity of these latter Weights. Now this Centre is an invariable Point as to Situation, because the Weights in F, G, H , viz. a, b, c are constant straight Lines, being the same in all Positions of the Point M . Therefore the Curve AMB must be such, that all the Perpendiculars to it do cut each other in one Point; that is, it is a Circle having that Point for its Centre. From whence we have the following remarkable Property of a Circle.

If there be ever so many Weights a, b, c , on the same Plane situate in F, G, H , and a Circle AMB be described about their common Centre of Gravity. And if the right Lines MF, MG, MH , &c. be drawn from any Point M in it: I say, the Sum of their Squares, each multiplied by the correspondent Weight, will always be equal to the same Quantity.

EXAMPLE II.

34. **L**ET the Curve AMB be such, that the FIG. 23.
right Line MF being drawn from any Point M in it to the Focus F of a standing Position, and the Perpendicular MG to the
Focus

Focus G , taken as a straight Line; the Relation of MF to MG will be always the same, as of the given Quantity a to the given one b .

Call FM, x ; and GM, y ; then will $x:y::a:b$. And therefore $ay = bx$. And this thrown into Fluxions will be $ay - bx = 0$. Now conceiving the Weight b in C taken on the other Side M with respect to F , and the Weight a in D (at a like Distance from M) and drawing the Line MP through the common Centre of Gravity of those Weights; and it will be the Perpendicular requir'd.

For from the Nature of the Balance it is plain, if the Cord CD be so divided in P that $CP:DP::a:b$, and the Point will be the common Centre of Gravity of the Weights in C and D .

The Curve AMB is a Conick Section; viz. a Parabola, when $a = b$; an Hyperbola, when a exceeds b ; and an Ellipsis, when the same is less.

EXAMPLE III.

FIG. 24. 35. IF you fasten the Ends of a Thread $FZVMGMXYH$ in the Points F and H , fixing a little Peg in the Point G , and then extend the Thread by means of the Pin M so, that the Parts FZV, HVX wrap about the Curves beginning in F and H ; and if the Part MG be doubled or folded together, the Motion of the Pin M will describe a Curve AMB . Now it is required to draw a Perpendicular MP to this Curve from a given Point M in it. The Position of the Thread aforesaid being given in that Point.

Here

Here I observe, that the straight Parts MF, MX of the Thread are always Tangents in V and X , and if the mixed Lines $FZVM$, and $HYXM$ be called x and z , the right Line MG, y ; and a right Line equal to the Length of the Thread, a . Then we shall always have $x + 2y + z = a$: whence we know that the Curve AMB is contained under the general Construction. Therefore finding the Fluxion of the Equation, viz. $\dot{x} + 2\dot{y} + \dot{z} = 0$, and conceiving the Weights 1 in C , 2 in D , and 1 in E . I say, the Line MP passing through the common Centre of Gravity of those Weights, will be the Perpendicular requir'd.

P R O P. XI.

36. **L**ET APB, EQF be any two Lines, FIG. 25. the Method of drawing Tangents PG, QH , to the same being known; and let PQ be a right Line, having a Point M given in it. Now if the Ends P and Q of this right Line move along the Lines AB, EF ; it is plain that the Point M by that Motion will describe a Curve CD . It is requir'd to draw a Tangent MT to the same from any given Point M .

Imagine the moveable right Line PMQ to come to a Situation infinitely near pmq , and draw the small right Lines PO, MR, QS , perpendicular to PQ , and we shall have the small right Angles pOP, mRM, qSQ ; likewise take $PK = MQ$, and draw the right Line HKG perpendicular to PQ , and continue out OP to T , whereat it is supposed to intersect the Tangent sought MT . This being done, it is evident that the little right Lines

Lines Op , Rm , Sq , will be equal to each other, because by Construction PM and MQ are every where the same.

Call the known Quantities PM or KQ , a ; MQ or PK , b ; KG , f ; KH , g ; and the little right Line Op , Rm , or Sq , y . Then the similar Triangles PKG and pOP , QKH and qSQ will give $PK(b):KG(f)::pO(y):OP = \frac{fy}{b}$. And $QK(a):KH(g)::qS(y):SQ = \frac{gy}{a}$. Now from common Geometry we know

$$\text{that } MR = \frac{OP \times MQ}{PQ} = \frac{QS \times PM}{PQ} = \frac{fy + gy}{a + b}.$$

So the similar Triangles mRM , MPT will give $mR(y):RM\left(\frac{fy + gy}{a + b}\right)::MP(a):PT = \frac{af + ag}{a + b}$. Which was to be found.

PROP. XII.

FIG. 26. 37. **L**ET BN, FQ be any two Lines, the right Lines BC, ED , cutting each other at right Angles in the Point A , being the Axes of them; and let LM be a Curve such, that the right Lines MGQ, MPN drawn from any Point M taken in it, parallel to AB, AE , the Relation of the Spaces $EGQF$, (the Point E being a Stable, one in the right Line AE , and the Line EF parallel to AC) $APND$, and the right Lines AP, PM, PN, GQ , be expressed by a given Equation. It is required to draw the Tangent MT to the given Point M in the Curve LM .

Call

Call AP or GM, x ; PM or AG, y ; PN, u ; GQ, z ; the Space $EGQF, s$; the Space $APND, t$; and the given Subtangents PH, a ; GK, b . Then will Pp or NS or $MR = \dot{x}$, Gg or Rm or $OQ = -\dot{y}$; $Sn = -\dot{u} = \frac{u\dot{x}}{a}$ because of the similar Triangles HPN, NSn ; $OQ = \dot{z} = -\frac{z\dot{y}}{b}$, $NPpn = \dot{t} = u\dot{x}$, and $QGgq = \dot{s} = -z\dot{y}$; where we are to take notice that the Values of Rm and Sn are negative, because when $AP(x)$ increases, $PM(y)$ and $PN(u)$ do decrease. This being supposed, throw the given Equation into Fluxions, wherein substitute $u\dot{x}$, $-z\dot{y}$, $-\frac{u\dot{x}}{a}$, $\frac{z\dot{y}}{b}$ for their Equals $\dot{t}$, $\dot{s}$, $\dot{u}$ and $\dot{z}$, and we shall have a new Equation expressing the Relation (sought) of $\dot{y}$ to $\dot{x}$, or MP to PT .

EXAMPLE I.

38. **L**ET $s + z z$ be $= t + u x$; this thrown into Fluxions is $\dot{s} + 2z\dot{z} = \dot{t} + u\dot{x} + x\dot{u}$, and putting for $\dot{s}, \dot{t}, \dot{z}, \dot{u}$ their Equals, we shall find $-z\dot{y} - \frac{2z z \dot{y}}{b} = 2u\dot{x} - \frac{u x \dot{x}}{a}$; from whence we get $PT\left(\frac{y\dot{x}}{\dot{y}}\right) = \frac{2ayzz + aybz}{bax - 2abu}$.

EXAMPLE II.

39. **L**ET s be $= t$: then $\dot{s} = \dot{t}$; that is, $-z\dot{y} = u\dot{x}$; and therefore $PT\left(\frac{y\dot{x}}{\dot{y}}\right) = -\frac{yz}{u}$. Now since this Quantity is negative, there-

* Art. 10. therefore the Point T must * be taken on the other Side the Point A , where the x 's begin. If we suppose the Line FQ to be an Hyperbola, whose Asymptotes are AC , AE , so that $GQ(z) = \frac{cc}{y}$, and the Line BND a straight Line parallel to AB ; whence $PN(u)$ will be always equal to the given right Line c . Then it is plain that the right Line AB will be an Asymptote to the Curve LM , and the Subtangent $PT\left(-\frac{yz}{u}\right) = -c$, that is, equal to a standing Quantity.

And so in this Case LM will be the Logarithmetick Curve.

PROP. XIII.

FIG. 27. 40. **L**ET BN , FQ be any two Lines, the right Line BA being their Axis, in which take the two Points A and E ; and let LM be a Curve such, that drawing a right Line AN from any Point M taken in it, describing the circular Arch MG from the Centre A , and drawing the Line G (parallel to EF) perpendicular to AB , the Relation of the Spaces $EGQF$, s ; ANB , t ; and the right Lines AM or AG , y ; AN , z ; GQ , u ; is expressed by any given Equation. It is required to draw the Tangent MT from a given Point M in the Curve LM .

Draw the right Line ATH perpendicular to AMN , and conceive another right Line Amn infinitely near AMN , another Arch mg , another Perpendicular gg , and the little Arch NS described from the Centre A . Now
2 call

call the given Subtangents AH, a ; and GK, b ; then will Rm or $Gg = \dot{y}$, $Sn = \dot{z}$: And because of the similar Triangles HAN, NSn ;

KGQ and QOq ; therefore $SN = \frac{a\dot{z}}{z}$, Oq

$= -\dot{u} = \frac{u\dot{y}}{b}$, $GQg = -\dot{s} = u\dot{y}$, ANn or

$AN \times \frac{1}{2} NS = -\dot{i} = \frac{1}{2} a \dot{z}$. All these Values must be put in the Fluxion of the given Equation, with which a new one will be formed; from whence we get a Value of $\dot{z}$ in $\dot{y}$. Now because of the Sectors and similar Triangles ANS and AMR , mRM and MAT , $AN(z)$

$: AM(y) :: NS\left(\frac{a\dot{z}}{z}\right) : MR = \frac{ay\dot{z}}{zz}$. And

$mR(\dot{y}) : RM\left(\frac{ay\dot{z}}{zz}\right) :: AM(y) : AT =$

$\frac{ayyz}{zz\dot{y}}$. Now if instead of $\dot{z}$ you substitute its Equal in $\dot{y}$, the Fluxions will vanish, and the Quantity of the Subtangent AT sought, will be expressed in known Terms. Which was to be found.

EXAMPLE I.

41. **L**ET $uy - s$ be $= z - t$; this thrown into Fluxions, is $uy + y\dot{u} - \dot{s} = z\dot{z} - \dot{t}$; and substituting as necessary we have $\dot{z} = \frac{4buy - 2uy\dot{y}}{4bz + ab}$. And lastly, putting this Va-

lue in $\frac{ayyz}{zz\dot{y}}$, there arises $AT = \frac{4abuyy - 2aay^3}{4bz^3 + abzz}$.

EXAMPLE

EXAMPLE II.

42. **L**ET $s=2t$; then $\dot{s}=2\dot{t}$, viz. $-u\dot{y}=-$
 az , or $\dot{z}=\frac{u\dot{y}}{a}$; therefore $AT\left(\frac{ayyz}{zz\dot{y}}\right)$
 $=\frac{uyy}{zz}$.

If the Line BN be a Circle, the Point A being the Centre, and the Radius be $AB=AN=c$; and if FQ be an Hyperbola such, that $GQ(u)=\frac{ff}{y}$; then it is manifest, that the

Curve LM makes an infinite Number of Revolutions about the Centre A before it comes to the same; because the Space FEQ becomes infinite when the Point G falls in A , and AT is $=\frac{ffy}{cc}$. Whence we may observe, that the Ratio of AM to AT is constant; and therefore the Angle AMT is a standing Quantity E ; and so the Curve LM is the *Logarithmical Spiral*.

PROP. XIV.

FIG. 28. 43. **L**ET AMD , BMC be any two Curves touching one another in the Point M , and let L be a given stable Point in the Plane of the Curve BMC . Now if you conceive the Curve BMC to revolve on (or roll along) the Curve AMD so, that the revolving Parts AM , BM , be always equal to each other. Now it is manifest that the Point L , moving along in the Plane BMC , will describe a kind of Cycloid ILK . This being premised, I say, if the right Line LM be drawn from the describing Point L ,

L, to the Point of Contact M in every different Position of the Curve BMC, the same will be perpendicular to the Curve ILK.

For the infinitely small equal Parts Mm , Mm of the Curves AMD , BMC , may be taken * for two little right Lines making an infinitely small Angle at the Point M . Now that the little Side Mm of the Curve or Polygon BMC , may fall upon, or coincide with the little Side Mm of the Curve or Polygon AMD ; it is necessary for the Point L to describe a little Arch Ll , about the Point of Contact M , as a Centre; and consequently the said small Arch will be a part of the Curve ILK ; and the right Line ML , which is perpendicular to it, will also be perpendicular to the Curve (ILK) in the Point L . Which was to be demonstrated.

PROP. XV.

44. **L**ET MLN be any right-lined Angle, FIG. 29. whose Sides LM , LN , touch any two Curves AM , BN . If these Sides move along the Curves, so as to touch them continually, it is evident, that L the Vertex of the Angle will describe the Curve ILK . Now it is requir'd to draw LC perpendicular to this Curve, the Position of the Angle MLN being given.

Describe a Circle thro' the Vertex L , and the Points of Contact M , N , and draw the right Line CL thro' the Centre of the same; I say, it will be perpendicular to the Curve ILK .

E

For

For conceiving the Curves AM , BN , as Polygons of infinite Numbers of Sides, of which Mm and Nn are each one; it is plain that if the Sides LM , LN , of the right-lined Angle LMN , of a given Quantity, move about the stable Points M , N , (the Tangents LM , LN , being supposed the Continuations of the little Sides Mf , Ng) until LM , the Side of the Angle, falls on the little Side Mm of the Polygon AM , and the other Side LN upon the little Side Nn of the Polygon BN ; the Vextex L will describe a little Part Ll of the circular Arch MLN . Therefore the said small Part Ll will be common to the Curve ILK ; and consequently the right Line CL , which is perpendicular to it, will be perpendicular to the Curve in the Point L . Which was to be demonstrated.

PROP. XVI.

FIG. 30. 45. **L**ET $ABCD$ be a flexible Cord having different Weights A , B , C , &c. hung to it at any given Distances AB , BC , &c. Now if this Cord be drawn along an Horizontal Plane by one End D thereof, which End moves along a given Curve DP in the said Horizontal Plane, it is manifest, that the said Weights will stretch the Cord while it is drawing along, and will describe the Curves AM , BN , CO , &c. It is required to draw Tangents to them, the Magnitude of the Weights, and the Position of the Cord $ABCD$ being given.

In the first Particle of Time, the End D moves forwards towards P , the Weights A ,
 B , C ,

$B, C,$ will describe, or endeavour to describe the little Sides $Aa, Bb, Cc,$ of Polygons, which the Curves $AM, BN, CO,$ are suppos'd to be; and therefore to draw the Tangents $AB, BG, CK,$ to them, is only the Determination of the Weights $A, B, C,$ in the said first Instant or Particle of Time, or the Position of the right Lines they endeavour to describe: To find which, we must observe,

1^o, That the Weight $A,$ in the first Particle of Time, is drawn in the Direction $AB;$ and since there is nothing to divert it from that Direction, because it draws no other Weight itself, it must keep to that Direction; and so the right Line AB will touch the Curve AM in $A.$

2^o, That the Weight B is drawn according to the Direction $BC;$ but because it draws the Weight A after it, which does not move in that Direction, and so must induce some Alteration to that Direction; the Direction of B will not be in the Line $BC,$ but in another Line $BG,$ whose Position may be thus found.

Describe a Rectangle $EF,$ with BC for the Diagonal, and having one Side BF in the Continuation of $AB;$ then if the Force wherewith the Weight B is drawn according to the Direction $BC,$ be expressed by $BC;$ it follows by the Laws of Mechanicks, that the Force BC may be divided into two others BE and $BF,$ viz. when the Weight B is drawn by the Force BC according to the Direction $BC,$ it is the same, as if it was drawn at the same time by the Force BE in the Direction $BE,$ and by the Force BF in the Direction $BF.$ Now the Weight A gives no Disturbance to the Direction $BE,$ it being drawn perpendicular

to it; and consequently the Force BE in that Direction will receive no Alteration; but it opposes the Force BF in the Direction BF by the whole Weight thereof. Therefore in order for the Weight B with the Force BF to overcome the Resistance of the Weight A , the Force BF must be divided into two Parts, having the same Proportion to each other, as the Weights A and B . Whence divide EC in the Point G so, that CG be to GE , as the Weight A to the Weight B ; then it is plain that EG will express the remaining Force wherewith the Weight B endeavours to move in the Direction BF , when it has overcome the Resistance of the Weight A . Therefore the Weight B is drawn in the same Time by the Force BE in the Direction BE ; and by the Force EG in the Direction BF or EC ; and so it will endeavour to move along BG with the Force BG : That is, BG will be the Direction, and consequently will touch the Curve BN in B .

3°. To find the Tangent CK . With CD as a Diagonal, make the Rectangle HI , the Side CI being in BC continued. Now the Weight B does not at all disturb the Force CH , wherewith the Weight C is drawn in the Direction CH ; but the Force CI in the Direction CI , is disturbed the greatest possible by the Weight B , and in some measure by the Weight A also. To find out the Quantity of these, draw AL perpendicular to BC continued out. (Here we may observe, that if AB expresses the Force wherewith the Weight A is drawn according to the Direction AB ; BL will express the Force wherewith the said Weight A is drawn in the Direction BC .) So that

that the Weight C , together with the Force CI , must overcome the whole Weight B , together with a Part of the Weight A , which is to the Weight A as BL to BA , or BF to BC . Therefore if $B + \frac{A \times BF}{BC} : C :: DK :$

KH , it is plain that CK will be the Direction of the Weight C , and consequently will be a Tangent to the third Curve CO in the Point C .

If there were a greater Number of Curves, the Tangents to the fourth, fifth, &c. might have been found after the same way; and the Tangents of the Curves described by the intermediate Points between the Weights, may be found by *Art.* 36.





S E C T. III.

Of the Use of Fluxions in finding the greatest and least Ordinates in a Curve, to which the Solution of Problems de MAXIMIS & MINIMIS may be reduced.

D E F I N I T I O N I.

FIG. 31, **L**ET MDM be a Curve, whose Ordinate
 32, PM, ED , PM are parallel to each
 33, other; and let this Curve be such, that while
 34, the Absciss AP continually increases, the Ordinate PM increases likewise, until it comes to a certain Point E , and afterwards decreases; or, on the contrary, if the same decreases until it comes to a certain Point E , and afterwards increases.

Then the Line ED is called the *greatest or least Ordinate, or a maximum or minimum*.

D E F I N. II.

IF a Quantity, as PM , be proposed, consisting of one or more indeterminate Quantities, as AP ; and while AP continually increases, the said Quantity PM increases likewise until it comes to a certain Point E , after which it constantly decreases, or contrariwise; and

and if it be required to find such a Value or Expression of AP , that the Quantity ED , of which it consists, may be greater or less than any other Quantity PM formed in like manner from AP . This is called a Problem *de maximis and minimis*.

GENERAL PROPOSITION.

46. **T**HE Nature of the Curve MDM being given: to find AE such a Value of AP , that the Ordinate ED be greater or lesser than any other PM of the same nature.

When AP , PM increase together, it is evident * that Rm the Fluxion of Pm will be * *Art. 8.* positive with regard to the Fluxion of AP . ^{10.} And on the contrary, if M decreases while the Ordinate AP increases, the Fluxion of PM will be negative. Now every Quantity that continually increases or decreases cannot from being positive become negative, without first passing thro' Infinity or nothing; *viz.* thro' 0 when the Quantity in the Beginning constantly goes on decreasing, and thro' Infinity when it continually increases in the Beginning. Therefore the Fluxion of a Quantity expressing a *maximum* or *minimum*, must be equal to 0, or Infinity. Now because the Nature of the Curve MDM is given, we can find (by *Seet. 1* or *2.*) a Value of Rm , which being first made equal to 0, and afterwards to Infinity; from thence in both the Suppositions the requir'd Value of AE will be had.

S H O L I U M.

FIG. 31, 32. 47. **T**HE Tangent in D is parallel to the Axis AB , when the Fluxion Rm becomes 0 in that Point: But when it becomes infinite, the Tangent coincides with the Ordinate ED . Whence we may observe, that the Ratio of mR to RM , viz. that of the Ordinate to the Subtangent in the Point D , is 0 or Infinite.

It easily appears that a Quantity continually decreasing, from positive cannot become negative without first passing thro' 0; but that a Quantity continually increasing must pass thro' Infinity to become negative, does not so easily appear. And therefore to assist the Imagination, let Tangents be conceived to issue from the Points M, D, M ; now in Curves, where the Tangent in D is parallel to the Axis AB , it is manifest that the Subtangent PT increases so much the more, as the Points M and P accede to D, E ; and so when M coincides with D , the same becomes infinite; and when at length AB is greater than AE , the Subtangent PT from positive becomes * negative, or contrariwise.

FIG. 31, 32.

* Art. 10.

EXAMPLE I.

FIG. 35. 48. **L**ET $x^3 + y^3 = axy$ (AP being $=x$, $PM=y$, $AB=a$) express the Nature of the Curve MDM . The Fluxion of the same will be $3xx\dot{x} + 3yy\dot{y} = ax\dot{y} + ay\dot{x}$, and $\dot{y} = \frac{ay\dot{x} - 3xx\dot{x}}{3yy - ax} = 0$, when the Point P coincides with the Point E sought. Whence we

we get $y = \frac{2x^2}{a}$. And putting this latter Part of the Equation for y in the Equation of the Curve $x^3 + y^3 = axy$, and there will arise $AE(x) = \frac{1}{3} a \sqrt[3]{2}$. Being such that the Ordinate ED will be a *maximum*, or the greatest of any other Ordinate PM to the Diameter AB .

EXAMPLE II.

49. **L**ET $y - a = a^{\frac{1}{3}} \sqrt[3]{a - x^{\frac{2}{3}}}$, express the Nature of the Curve MDM . This FIG. 33.
thrown into Fluxions will be $\dot{y} = \frac{-2 \dot{x} \sqrt[3]{a}}{3 \sqrt[3]{a - x}}$,
which I first make equal to 0; but because this Supposition gives us $-2 \dot{x} \sqrt[3]{a} = 0$, from which the Value of AE cannot be known, I afterwards make $\frac{-2 \dot{x} \sqrt[3]{a}}{3 \sqrt[3]{a - x}}$ infinite; and so $3 \sqrt[3]{a - x} = 0$. Consequently $AE(x) = a$.

EXAMPLE III.

50. **L**ET AMF be a Semicycloid, whose FIG. 36.
Base BF is less than half the Circumference ANB of the generating Circle, and Centre is C . It is required to find the Point E in the Diameter AB being such, that the Ordinate ED shall be a *maximum*, or the greatest possible.

Draw the Ordinate PM at pleasure, intersecting the Semicircle in N , and at the Points M, N of the Ordinate conceive the little Triangles MRm, NSn ; and calling the indeterminate Quantities AP, x ; PN, z ; the Arch AN, u ;

AN, u ; and the given Quantities ANB, a ; BF, b ; CA or CN, c . Then from the Nature of the Cycloid $ANB(a):BF(b)::AN(u):$
 $NM = \frac{bu}{a}$. Whence $PM = z + \frac{bu}{a}$, and

the Fluxion thereof $Rm = \frac{az + bu}{a} = 0$,

when the Point P coincides with E the Point sought. Now the right-angled Triangles NSn, NPC are similar. For if the common Angle CNS be taken away from the right Angles CNn, PNS the remaining Angles SNn, PNC shall be equal. And therefore

$$CN(c):CP(c-x)::Nn(u):Sn(z) = \frac{cu-xu}{c}.$$

Whence putting this Value for z in $az + bu = 0$,

and there will arise $\frac{acu-axu+bcu}{c} = 0$; and

so we get x (which in this Case is AE) = $c + \frac{bc}{a}$.

Therefore assume CE towards B , a fourth Proportional to the Semi-circumference ANm , the Base BF , and the Radius CB , and the Point E will be that sought.

EXAMPLE IV.

FIG. 35. 51. **T**O cut or divide the given right Line AB in the Point E so, that the Product of the Square of AE , one of the Parts into the other Part, be the greatest of any Product of the like nature.

Call the unknown Quantity AE, x ; and the given Quantity AB, a ; then must $AE \times EB = ax - x^2$ be a *maximum*. Now a Curve
 MDM

of FLUXIONS.

59

MDM must be supposed such, that the Relation of the Ordinate $MP(y)$ to the correspondent Absciss $AP(x)$ is expressed by $y = \frac{axx - x^3}{aa}$, and the Point E must be found

such, that the Ordinate ED be a *maximum*; and so $\dot{y} = \frac{2ax\dot{x} - 3xxx}{aa} = 0$, from whence we

get $AE(x) = \frac{2}{3}a$.

And generally if you would have $x^m \times \overline{a-x}^n$ (where m and n express any Numbers at pleasure) to be a *maximum*, the Fluxion of it must be made equal to 0, or Infinity. Whence $m x^{m-1} \dot{x} \times \overline{a-x}^n - n a \overline{a-x}^{n-1} \dot{x} \times x^m = 0$; which divided by $x^{m-1} \times \overline{a-x}^{n-1} \dot{x}$, and there comes out $am - mx - nx = 0$, and $AE(x)$

$$= \frac{m}{m+n} a.$$

If $m=2$, and $n=-1$; then will $AE=2a$, and the Problem may be thus laid down.

Continue out the given Line AB (towards FIG. 37. B) to the Point E , in such manner that

the Quantity $\frac{AE^2}{BE}$ be a *minimum*, and not a *maximum*; for the Equation of the Curve MDM will be $\frac{xx}{x-a} = y$, wherein if we suppose $x=a$, the Ordinate PM , which becomes BC , will be $\frac{aa}{0}$, that is, infinite; and conceiving x infinite one shall have $y=x$, viz. the Ordinate will be also infinite.

If $m=1$, and $n=-2$; then will $AE=-a$; whence the Problem may be after this manner stated.

Con-

FIG. 38. Continue out the given right Line AB (towards A) to the Point E so, that the Quantity $\frac{AE \times AB}{BE}$ be the greatest of any other the like Quantity $\frac{AP \times AB}{BP}$.

EXAMPLE V.

FIG. 39. 52. THE right Line AB being divided into three Parts AC, CF, FB . To divide or cut the middle Part CF in the Point E being such, that the Ratio of the Rectangle $AE \times EB$ to the Rectangle $CE \times EF$, be less than any other Ratio formed in like manner.

Call the given Quantities AC, a ; CF, b ; CB, c ; and the unknown Quantity CE, x : then will $AE = a + x$, $EB = c - x$, $EF = b - x$, and therefore the Ratio of $AE \times EB$, to $CE \times EF$ will be $\frac{ac + cx - ax - xx}{bx - xx}$, which

must be a *minimum*. Whence if a Curve MDM be supposed such, that the Relation of the Ordinate $PM(y)$ to the Absciss $CP(x)$ be expressed by $y = \frac{aac + acx - aax - axx}{bx - xx}$,

the Problem will be brought to this, *viz.* to find such a Value CE for x , that the Ordinate ED be less than any other PM of like sort. Therefore throwing the said Equation into Fluxions, and afterwards dividing by $a\dot{x}$, there will come out $cxx - axx - bxx + 2acx - abc = 0$; and one Root of this Equation will solve the Problem.

If $c = a + b$; then will $x = \frac{1}{2}b$.

EXAM.

EXAMPLE VI.

53. **O**F all the Cones that can be inscribed within a given Sphere, to find that whose convex Surface is the greatest. FIG. 40.

This Problem, in other Words, is to find the Point E in AB the Diameter of the Semicircle AFB such, that drawing the Perpendicular EF , and joining AF , the Rectangle $AF \times FE$ be greater than any other Rectangle ($AN \times NP$) like it. For if the Semicircle AFB be conceived to make an entire Revolution about the Diameter AB , it is evident, that it shall describe a Sphere, and the right-angled Triangles AEF , APN , will generate Cones inscribed in the Sphere; the convex Surfaces of which described by the Chords AE , AN , will be to one another as the Rectangles $AF \times FE$, $AN \times NP$.

—Now let the unknown Quantity $AE = x$, and the given one $AB = a$. Then from the Nature of the Circle $AF = \sqrt{ax}$, $EF = \sqrt{ax - xx}$; and therefore $AF \times FE = \sqrt{aaxx - ax^3}$ which must be a *Maximum*. Whence we must conceive a Curve MDM to be such, that the Relation of the Ordinate PM (y) to the correspondent Absciss AP (x) is expressed by $\frac{\sqrt{aaxx - ax^3}}{a} = y$; and find the Point E so,

that the Ordinate ED be greater than any other (PM) of like sort. And making the

Fluxion of the Equation $\frac{2ax\dot{x} - 3xx\dot{x}}{2\sqrt{aaxx - ax^3}} = 0$, we get $AE(x) = \frac{2}{3}a$.

EXAMPLE

EXAMPLE VII.

54. **A**MONG all the Parallelepipedons equal to a given Cube a^3 , and having the given right Line b for one of the Sides, to find that which has the least Superficies.

Call one of the two Sides sought x , and the other will be $\frac{a^3}{bx}$; then assuming the alternate

Planes $b, x, \frac{a^3}{bx}$ of the Parallelepipedon, and their Sum, viz. $b + x + \frac{a^3}{x} + \frac{a^3}{b}$ will be half

of the Superficies, which must be a *Minimum*. And so (as all along) conceiving a Curve expressed by $\frac{bx}{a} + \frac{aa}{x} + \frac{aa}{b} = y$, the Fluxion of this Equation must be equal to 0; that is $\frac{b\dot{x}}{a}$

$-\frac{aa\dot{x}}{xx} = 0$, from whence there comes out

$xx = \frac{a^3}{b}$, and $x = \sqrt{\frac{a^3}{b}}$: Consequently the three

Sides of the Parallelepipedon required, will be

$b, \sqrt{\frac{a^3}{b}}$, and $\sqrt{\frac{a^3}{b}}$. Whence you may observe,

that two Sides are equal.

EXAMPLE VIII.

FIG. 41. 55. **A**MONG all the Parallelepipedons that are equal to a given Cube a^3 , to find that which has the least Superficies.

Call one of the unknown Sides x ; then by the last Example it is plain, that the two other

ther Sides will be each equal to $\frac{\sqrt{a^3}}{b}$; and therefore the Sum of the alternate Planes, which is the half of the Superficies, will be $\frac{a^3}{x} + 2\sqrt{a^3x}$, which must be a *Minimum*. Whence the Fluxion of this must be equal to 0, viz. $\frac{a^3\dot{x}}{xx} + \frac{a^3\dot{x}}{\sqrt{a^3x}} = 0$; and so $x = a$; and consequently the two Sides shall also be equal to a ; so that the Cube itself solves the Problem.

EXAMPLE IX.

56. **T**HE Line AEB being given in Position FIG. 413 on a Plane, together with two stable Points C, F ; and if two right Lines $CP (u)$, $PF (z)$, be drawn from any Point P in it; and if a Quantity be made up of these indeterminate ones u and z , and other given right Lines a, b , &c. at pleasure. It is required to find such a Position of the right Lines CE, EF , that the Quantity given, which is made up of them, be greater or lesser than that same Quantity when it is made up of the right Lines CP, PE .

Let us suppose the Lines CE, EF , to have the requisite Situation: Join CF , and conceive the Curve DM to be such, that drawing PQM at pleasure perpendicular to CF , the Ordinate QM may express the Quantity given. Now it is manifest, that when the Point P falls in the Point E , the Ordinate QM , which then becomes OD , must be the greatest or least of all others of like sort. Therefore the Fluxion of it must be equal to 0, or infinite:

nite: Whence if the given Quantity, for Example, is $au \times zz$; then will $a\dot{u} \times 2z\dot{z} = 0$, and consequently $\dot{u} : -\dot{z} :: 2z : a$. Wherefore we may already perceive that $\dot{z}$ must be negative with respect to $\dot{u}$; that is, the right Lines CE , EF , must have such a Position that z decreases at the same time as u increases.

Now if EG be drawn perpendicular to the Line AEB , and from any Point G therein, the Lines GL , GI , be likewise drawn perpendicular to CE , EF , and the right Lines CKe , FeH , be drawn from the Point e infinitely near E , and from the Centres C , F , be described the small Arches EK , EH ; the right-angled Triangles ELG and EKe , EIG and EHF , will be similar. For if the Angle LEe be taken from the right Angles $G\dot{E}e$, LEK , the remaining Angles LEG , KBe , shall be equal. Whence $GL : GI :: KB (\dot{u}) : He (-\dot{z}) :: 2z : a$. Therefore the Position of the right Lines CE , EF , must be such, that when EG is drawn perpendicular to the Line AEB , the Sine (GL) of the Angle GEC , is to the Sine (GI) of the Angle GEF , as the Quantities drawn into $\dot{z}$, to the Quantities drawn into $\dot{u}$. Which was to be found.

C O R O L.

57. **I**F the right Line CE be given both in Position and Magnitude, and EF only in Magnitude, and the Position thereof be required, it is evident that the Angle GEC being given, its Sine (GL) will be given likewise, and consequently the Sine (GI) of the Angle (GEF) sought. Therefore if a Circle
I be

be described with EG as a Diameter, and the Value of GI be laid off in the Circumference from G to I ; the right Line EF passing thro' the Point I , will have the requisite Position.

Let $au + bz$ be the given Quantity; then will GI be $= \frac{a \times GL}{b}$: and so it appears, that

let EC and EF have what Length soever, the Position of this latter shall be always the same, since they do not come into the Value GI , which consequently does not vary. If $a = b$, it is plain that the Position of EF must be in CE continued out from E ; because $GL = GI$, when the Points C and F fall on each Side the Line AB : But when they fall on the same Side, the Angle FEG must be assumed equal to the Angle CEG . FIG. 42.

EXAMPLE X.

58. **T**HE Circle AEB being given in Position, as also the Points C and F without the same: to find the Point E in the Periphery being such, that the Sum of the right Lines CE , EF may be a *minimum*. FIG. 42.

Suppose the Point E to be that sought, and draw the Line OEG from the Centre O ; which will be perpendicular to the Circumference AEB ; and so * the Angles FEG , CEG will be equal. Therefore if EH be drawn so, that the Angle EHO be equal to the Angle CEO , and likewise EK so, that the Angle EKO be equal to the Angle FEO , and the Parallels ED , EL to OF , OC ; and there will be formed the similar Triangles OCE and OEK , OFE and OEH , HDE and KLE ; and calling the known Quantities R * Art. 57.
 OE ,

OE , OA or OB, a ; OC, b ; OF, c ; and the unknown ones OD or LE, x ; DE or OL, y .

Then will $OH = \frac{aa}{b}$, $OK = \frac{aa}{c}$, and HD

$$\left(x - \frac{aa}{b}\right) : DE(y) :: KL \left(y - \frac{aa}{c}\right) : LE(x).$$

Whence $xx - \frac{aax}{b} = yy - \frac{aay}{c}$; and this

is an Equation appertaining to an Hyperbola, which may be easily constructed, and will cut the Circle in the Point E sought.

EXAMPLE XI.

FIG. 43. 59. **A** TRAVELLER setting out from a Place C to go to a Place F , must cross two Countries divided from each other by the right Line AEB . Now suppose him to go the Length a in the time c in the Country adjoining to C , and the Length b in the same time c in the other Country adjoining to F : it is required to find the Point E in the right Line AEB thro' which he is to pass in the shortest time possible from C to F .

Make $a : CE(u) :: c : \frac{cu}{a}$. And $b : EF(z) ::$

$c : \frac{cz}{b}$. Then it is plain, that $\frac{cu}{a}$ expresses the Time of the Travellers going the Length CE , and $\frac{cz}{b}$ the Time of his going the Length EF ; so

Art. 56. that $\frac{cu}{a} + \frac{cz}{b}$ must be a *minimum*. Whence * drawing EG perpendicular to AB , the Sine of the Angle GEC must be to the Sine of the Angle GEF , as a to b .

This

This being premised, if the Circle CGH be described with the Radius EC from the Point sought E as a Centre, and the Perpendiculars CA, HD, FB be drawn to the right Line AEB , and the Perpendiculars GL, GI to the right Lines CE, EF , we shall have $a:b::GL:GI$. But $GL=AE$, and $GI=ED$, because the right-angled Triangles GEL and ECA, GEI and EHD are equal and similar, as may be easily proved. Therefore if the unknown Line AE be called x , we shall have $ED = \frac{bx}{a}$: and calling the known Lines

AB, f ; AC, g ; BF, b : from the Similarity of the Triangles $EBF, EDH, EB(f-x)$:

$$BF(b)::ED\left(\frac{bx}{a}\right):DH=\frac{bbx}{af-ax}.$$

But because of the right-angled Triangles EDH, EAC , of equal Hypotenuses EH, EC , $ED^2 + DH^2$ will be $=EA^2 + AC^2$, that is,

$$\frac{bbxx}{aa} + \frac{bbbx}{aaff-2aafx+axxx} = xx + gg:$$

and freeing the Equation from Fractions, and afterwards duly ordering the same, there will arise

$$\begin{aligned} aax^4 - 2aafx^3 + aaffxx - 2aafggx + aaffgg &= 0. \\ -bb + 2bbf &+ aagg \\ &-bbff \\ &-bbbb \end{aligned}$$

This Equation may be gotten after the following manner, without having recourse to the 9th Example.

Having named the known Lines as before, viz. AB, f ; AC, g ; BF, b ; and the unknown one AE, x . Make $a:CE(\sqrt{gg+xx}):c:$
 $\frac{c\sqrt{gg+xx}}{a} =$ to the Time the Traveller is go-

ing the Length CE . And in like manner
 $b : EF (\sqrt{ff - 2fx + xx + bb}) :: c :$

$$\frac{c\sqrt{ff - 2fx + xx + bb}}{b} = \text{Time of his going}$$

the Length EF : so that $\frac{c\sqrt{gg + xx}}{a} +$

$$\frac{c\sqrt{ff - 2fx + xx + bb}}{b} = \text{a minimum: and there-}$$

$$\text{fore } \frac{cx\dot{x}}{a\sqrt{gg + xx}} + \frac{cx\dot{x} - cf\dot{x}}{b\sqrt{ff - 2fx + xx + bb}} = 0.$$

Whence dividing by $c\dot{x}$, and freeing the Equation from Surds, we shall have the same Equation as before; one Root of which will express AE the *minimum* sought.

EXAMPLE XII.

FIG. 34. 60. **L**ET F be a Pulley hanging freely from the End of a Cord CF fasten'd in C , and let D be a Weight suspended by the Cord DFB put over the Pulley F , which Cord is fasten'd in B ; so that the Points C and B lie in the same horizontal Line CB . Now if the Pulley and Cords be supposed to have no Weight, it is required to find in what Place the Weight D , or Pulley F , will settle or come to rest.

By the Principles of Mechanicks, it is plain that the Weight D will descend as far as possible below the horizontal Line CB : therefore the Plumb Line DFE will be a *maximum*. And therefore calling the given Quantities CF, a ; DFB, b ; CB, c ; and the unknown Quantity CE, x ; there will arise $EF = \sqrt{aa - xx}$, $FB = \sqrt{aa + cc - 2cx}$, and $DFE = b - \sqrt{aa + cc - 2cx} + \sqrt{aa - xx}$, which must be a *maximum*. And so the Fluation

xion of it will be $\frac{c\dot{x}}{\sqrt{aa+cc-2cx}} - \frac{x\dot{x}}{\sqrt{aa-xx}}$
 $= 0$. Whence there arises $2cx^3 - 2ccxx -$
 $aaax + aacc = 0$, and dividing by $x - c$,
 there comes out $2cxx - aax - aac = 0$, one
 Root of which will express CE such, that
 the Perpendicular ED passes by the Pulley F
 and the Centre D of the Weight, when the
 same is at rest.

Here follows another Solution of this Problem.

Call EF, y ; BF, z ; then will $b - z + y =$
maximum, and so $\dot{y} = \dot{z}$. Now it is evident,
 that the Pulley F does describe a Circle CFA
 about the Point C as a Centre: and if fR be
 drawn from the Point f , infinitely near to F ,
 parallel to CB , and fS perpendicular to BF ;
 therefore will $FR = \dot{y}$, and $FS = \dot{z}$. Which
 are consequently equal to each other: and so
 the little right-angled Triangles FRf , FSf ,
 having the common Hypothenuse Ff , are
 equal and similar: whence the Angle RFf is
 equal to the Angle SFf , that is, the Point F
 must be so situate in the Periphery FA , that
 the Angles made by the right Lines EF, FB ,
 with the Tangents in F , be equal to each o-
 ther: or else (which comes to the same) the
 Angles BFC, DFC equal.

This being premised, if you draw FH so,
 that the Angle FHC be equal to the Angle
 CFB or CFD ; the Triangles CBF, CFH
 will be similar; as also the right-angled Tri-
 angles ECF, EFH , since the Angle CFE is
 equal to the Angle FHE , each of them be-
 ing the Complement of the equal Angles FHC ,
 CFD to two right Angles; and consequently

F 3

CH

ing the Length CE . And in like manner
 $b : EF (\sqrt{ff - 2fx + xx + bb}) :: c :$

$$\frac{c\sqrt{ff - 2fx + xx + bb}}{b} = \text{Time of his going}$$

the Length EF : so that $\frac{c\sqrt{gg + xx}}{a} +$

$$\frac{c\sqrt{ff - 2fx + xx + bb}}{b} = \text{a minimum: and there-}$$

$$\text{fore } \frac{cx\dot{x}}{a\sqrt{gg + xx}} + \frac{cx\dot{x} - cf\dot{x}}{b\sqrt{ff - 2fx + xx + bb}} = 0.$$

Whence dividing by $c\dot{x}$, and freeing the Equation from Surds, we shall have the same Equation as before; one Root of which will express AE the *minimum* sought.

EXAMPLE XII.

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By the Principles of Mechanicks, it is plain that the Weight D will descend as far as possible below the horizontal Line CB : therefore the Plumb Line DFE will be a *maximum*. And therefore calling the given Quantities CF, a ; DFB, b ; CB, c ; and the unknown Quantity CE, x ; there will arise $EF = \sqrt{aa - xx}$, $FB = \sqrt{aa + cc - 2cx}$, and $DFE = b - \sqrt{aa + cc - 2cx} + \sqrt{aa - xx}$, which must be a *maximum*. And so the Fluxion

xion of it will be $\frac{c\dot{x}}{\sqrt{aa+cc-2cx}} - \frac{x\dot{x}}{\sqrt{aa-xx}}$
 $= 0$. Whence there arises $2cx^3 - 2ccxx -$
 $aa\dot{x}x + aacc = 0$, and dividing by $x - c$,
 there comes out $2c\dot{x}x - aax - aac = 0$, one
 Root of which will express CE such, that
 the Perpendicular ED passes by the Pulley F
 and the Centre D of the Weight, when the
 same is at rest.

Here follows another Solution of this Problem.

Call EF, y ; BF, z ; then will $b - z + y =$
maximum, and so $\dot{y} = \dot{z}$. Now it is evident,
 that the Pulley F does describe a Circle CFA
 about the Point C as a Centre: and if fR be
 drawn from the Point f , infinitely near to F ,
 parallel to CB , and fS perpendicular to BF ;
 therefore will $FR = \dot{y}$, and $FS = \dot{z}$. Which
 are consequently equal to each other: and so
 the little right-angled Triangles FRf, FSf ,
 having the common Hypothenuse Ff , are
 equal and similar: whence the Angle RFf is
 equal to the Angle SFf , that is, the Point F
 must be so situate in the Periphery FA , that
 the Angles made by the right Lines EF, FB ,
 with the Tangents in F , be equal to each o-
 ther: or else (which comes to the same) the
 Angles BFC, DFC equal.

This being premised, if you draw FH so,
 that the Angle FHC be equal to the Angle
 CFB or CFD ; the Triangles CBF, CFH
 will be similar; as also the right-angled Tri-
 angles ECF, EFH , since the Angle CFE is
 equal to the Angle FHE , each of them be-
 ing the Complement of the equal Angles FHC ,
 CFD to two right Angles; and consequently

F 3

CH

$$CH = \frac{aa}{c}, \text{ and } HE \left(x - \frac{aa}{c} \right) : EF(y) : EF(y) ::$$

$$EC(x). \text{ Whence } xx - \frac{aax}{c} = yy = aa - xx \text{ from}$$

the Nature of the Circle: from whence arises the same Equation as at first.

EXAMPLE XIII.

FIG. 45. 61. **T**HE Elevation of the Pole being given: To find the Day of the Year wherein the Twilight will be the shortest possible.

Let C be the Centre of the Sphere; $APTOBHQ$ the Meridian; $HD\theta O$, the Horizon; $Q\acute{E}eT$ the Crepuscular Circle parallel to the Horizon; $AMNB$ the Equator; $FEDG$ that Part of the Parallel to the Equator (the Sun describes the Day wherein the Twilight is shortest) contained between the Planes of the Horizon and the Crepuscular Circle; P the South Pole; $P\acute{E}M$, $P\acute{D}M$ Quadrants of Circles of Declination. Now the Arch HQ or OT of the Meridian comprehended between the Horizon and the Crepuscular Circle, and the Arch OP of the Elevation of the Pole are given: and consequently their Sines CI or FL or QX , and OV . Now we must find CK the Sine of the Arch EM or DN of the Sun's Declination, when he describes the Parallel ED .

If you suppose another Part $fedg$ of a Parallel to the Equator infinitely near to $FEDG$, and draw the Quadrants Pem , Pdn : it is manifest, that the Time the Sun takes up in describing the Arch ED must be a *minimum*, and the Fluxion of the Arch MN being the Measure of it, and becoming mn when ED be-

comes ed , must be equal to nothing; whence the small Arches Mm , Nn , and consequently the little Arches Re , Sd , will be equal to each other. Now the Arches RE , SD being contained between the same Parallels ED , ed , are equal likewise, and the Angles at S and R are right ones. Therefore the little right-angled Triangles ERe , DSd (consider'd as right * lin'd Triangles, on account of their * *Art. 3.* Sides being infinitely small) will be equal and similar: and consequently the Hypothenuses Ee , Dd shall be equal likewise.

This being premised, the right Lines DG , EF , dg , ef , the common Sections of the Planes $FEDG$, $fedg$, parallel to the Equator, and the Horizon and Crepuscular Circle, will be perpendicular to the Diameters HO , QT , because the Planes of all these Circles are perpendicular to the Plane of the Meridian; and the little right Lines Gg , Ff , will be equal to each other, since the right Lines FG , fg are parallel. Therefore $\sqrt{Dd^2 - Gg^2}$, or $DG - dg = \sqrt{Ee^2 - Ff^2}$, or $fe - FE$. Now it is plain (from *Art. 50.*) if two Ordinates in a Semi-circle be drawn infinitely near, the small Arch intercepted between them will be to their Fluxion, as the Radius is to the Part or the Diameter intercepted by the Centre and those Ordinates. Whence (because of the Circles HDO , QET) $CO:CG::Dd$ or $Ee:DG - dg$ or $fe - FE::IQ:IF::CO + IQ$ or $OX:CG + IF$ or GL . But because of the right-angled similar Triangles CVO , CKG , FLG : therefore $CO:CG::OV:GK$. And $GK:GL::GK:FL$ or QX . Whence $OV:CK::OX:XQ::XQ:XH$ by the Nature of the

Circle: that is, if QX be taken for the Radius in the right-angled Triangle QXH , and the Angle HQX be 9 Degrees, (because the Arch HQ by Astronomers is supposed to be 18 Degrees) and you make as the Radius is to the Tangent of an Angle of 9 Degrees, so is the Sine of the Elevation of the Pole to a fourth Sine; this will be the Sine of the Sun's *Southern* Declination that Day of the Year the Twilight is the shortest possible. So that if you take 0.8002875 from the Logarithm of the Sine of the Elevation of the Pole, the Remainder will be the Logarithm of the Sine sought.





S E C T. IV.

Of the Use of Fluxions in finding the Points of Inflexion and Retrogression of Curves.

BECAUSE second, third, &c. Fluxions are used hereafter; before we go further, we think it necessary to define them.

D E F I N I T I O N I.

THE infinitely small Part generated by the continual increasing or decreasing of the Fluxion of a variable Quantity, is called the * *Fluxion of the Fluxion of that Quantity, or second Fluxion.* So if a third Ordinate nq be supposed infinitely near the second mp , and mS be drawn parallel to AB , and mH to RS . Hn is the *Fluxion of the Fluxion Rm , or the second Fluxion of PM .* FIG. 46.

In like manner, if a fourth Ordinate of be supposed infinitely near the third nq , and nT be drawn parallel to AB , and nL to ST , the Difference of the small right Lines Hn , Lo , is the *Fluxion of the second Fluxion, or the third Fluxion of PM .* Understand the same of others.

* See the Translator's Preface.

OBSERVATION.

Fluxions of Fluxions are denoted with Dots over the Letters, being so many in Number as the Order of the Fluxion is. For Example, $\ddot{x}$ is the second Fluxion of x ; $\dot{\dot{x}}$ the third Fluxion of x ; $\ddot{\dot{x}}$ the fourth Fluxion of x , &c. Understand the same of y , or any other variable Quantity. And so if $MP=y$, then will $\ddot{y}$ express Hn ; $\dot{\dot{y}}$, $Lo-Hn$ or $Hn-Lo$, &c.

COROLLARY I.

62. **I**F each of the Abscisses AP, Ap, Aq, Af , be called x , and each of the correspondent Ordinates PM, pm, qn, fo, y ; and every of the Parts of the Curve AM, Am, An, Ao, u ; then it is plain that $\dot{x}$ will be the Fluxions Pp, pq, qf , of the Abscisses; $\dot{y}$ the Fluxions Rm, Sn, To , of the Ordinates; and $\dot{u}$ the Fluxions Mm, mn, no , of the Parts of the Curve AMD . Now in order (for Example) to get the second Fluxion Hn of the variable Quantity PM , we must suppose two little Parts Pp, pq , in the Axis, and two others Mm, mn , in the Curve, to get the two Fluxions Rm, Sn ; and therefore if the two small Parts Pp, pq , be conceived as equal to one another, it is evident that $\dot{x}$ will be a standing Quantity with respect to $\dot{y}$ and $\dot{u}$, because Pp which becomes pq , will not vary while Rm , which becomes Sn , and Mm which becomes mn , does. If the little Parts Mm, mn , of the Curve, be supposed to be equal to each other, then will $\dot{u}$ be a standing Quantity with respect to $\dot{x}$ and $\dot{y}$. And lastly, if Rm and Sn be sup-

supposed equal, then will $\dot{y}$ be a standing Quantity with regard to $\dot{x}$ and $\dot{u}$, and the Fluxion Hn ($\ddot{y}$) of it will be equal to nothing.

In like manner, to get the third Fluxion of PM , or the Fluxion of the second Fluxion Hn , we must imagine three little Parts Pp , pq , qf , in the Axis; three others Mm , mn , no , in the Curve, and three others Rm , Sn , To , in the Ordinates: Then will $\dot{x}$, $\dot{u}$, or $\dot{y}$, be a constant or standing Quantity, according as the small Parts Pp , pq , qf , or Mm , mn , no , or Rm , Sn , To , be supposed equal. The same must be conceived of the fourth, fifth, &c. Fluxions.

All this is to be understood also in Curves, FIG. 47. as AMD , whose Ordinates BM , Bm , Bn , all issue from a stable Point B . As, for Example, to get the second Fluxion of BM , we must conceive two other Ordinates Bm , Bn , making infinitely small Angles MBm , mBn ; then if the small Arches of Circles MR , mS , be described from the Centre B , the Difference of the small right Lines Rm , Sn , shall be the second Fluxion of BM . And the small Arches MR , mS , or the small Parts Mm , mn , of the Curve, or, finally, the little right Lines Rm , Sn , may be taken as standing Quantities. Understand the same of third, fourth, &c. Fluxions of the Ordinate BM .

SCHOLIUM.

63. **H**ERE we are to observe, 1^o, That if FIG. 46.
there be several Orders of infinitely small Quantities: For Example, RM will be infinitely small with respect to PM , and infinitely great with respect to Nn . In like manner,

ner, the Space $MPpm$ is infinitely small with regard to the Space APM , and infinitely great with regard to the Triangle MRm .

2°, That the whole Fluxion Pf is likewise infinitely small with regard to AP ; because every Quantity which is the Sum of a finite Number of infinitely small Quantities, as Pp , pq , qf , with respect to another AP , will be infinitely small with regard to the said Quantity: And that in order for it to become of the same Order, it is necessary for the Number of Quantities of the Order next below it, and of which it consists, to be infinite.

COROL. II.

FIG. 48, 64. THE second Fluxions, in all the Cases possible, may be represented thus:

1°, In Curves whose Ordinates mR , nS , are parallel; continue out the small Side Mm to intersect the Ordinate Sn in H , and describe the Arch nk from the Centre m with the Distance mn , and draw the little right Lines nl , li , kc , parallel to mS and Sn . This being done, if $\dot{x}$ be supposed a standing Quantity, viz. $MR = mS$, it is plain that the Triangle mSH is similar and equal to the Triangle MRm ; and so Hn is $=\ddot{y}$; that is, the Difference of Rm and Sn , and $Hk = \ddot{u}$. But if $\dot{u}$ be conceived to be constant, viz. $Mm = mn$ or mk , then it is plain that the Triangle mgk is similar and equal to the Triangle MRm ; and so $kc = \ddot{y}$, and Sg or $cn = \ddot{x}$. Lastly, if $\dot{y}$ be supposed invariable, viz. $mR = nS$, then will the Triangle mil be equal and similar to the Triangle MRm ; and so iS or $nl = \ddot{x}$, and $lk = \ddot{u}$.

2°, In

2^o, In Curves whose Ordinates BM, Bm, Bn , issue from a Point B . Describe the Arches MR, mS , from the Centre B ; these may be looked * upon as small right Lines perpendicular to Bm, Bn . Continue out Mm to E , and describe the small Arch nkE from the Centre m , with the Distance mn ; make the Angle $EmH = mBn$, and draw the little right Lines nl, li, kcg parallel to mS and Sn . This being done, because the Triangle BSm is right-angled at S , the Angle $BmS + mBn$, or $+EmH =$ a right Angle; and therefore the Angle $BmE =$ a right Angle $+ SmH$: It is also equal to the right Angle $MRm + RMm$, since it is external to the Triangle RmM : Therefore the Angle $SmH = RMm$.

1. Hence, if you suppose $\dot{x}$ invariable, that is, the little Arches MR, mS , equal, the Triangle SmH will be similar and equal to the Triangle RMm , and so $Hn = \ddot{y}$, and $HK = \ddot{u}$.

2. If $\ddot{u}$ be supposed invariable, the Triangle gmK will be equal and similar to the Triangle RMm ; and so Kc will express $\ddot{y}$ and Sg or $cn, \ddot{x}$. Lastly, if $\ddot{y}$ be supposed invariable, the Triangles iml, RMm , will be equal and similar; and so iS or $ln = \ddot{x}$, and $lk = \ddot{u}$.

P R O P. I.

65. **T**O find the Fluxion of a Quantity consisting of Fluxions.

Consider any one of the Fluxions of the given Expression as invariable, and proceed with the others as variable Quantities, according to the Rules laid down in Section the first.

For

For Example, if x be supposed invariable, the Fluxion of $\frac{y\dot{y}}{x}$ will be $\frac{\dot{y}\dot{y}+y\ddot{y}}{x}$, and taking

$\dot{y}$ as invariable, it will be $\frac{\dot{x}\dot{y}-y\dot{y}\dot{x}}{xx}$. In like

manner, taking x as invariable, the Fluxion

of $\frac{z\sqrt{xx+yy}}{x}$, will be $\dot{z}\sqrt{xx+yy}+\frac{z\dot{y}\dot{y}}{\sqrt{xx+yy}}$,

and dividing it by $\dot{x}$, there will arise

$\frac{\dot{z}xx+\dot{z}y\dot{y}+\dot{z}\dot{y}\dot{y}}{\dot{x}\sqrt{xx+yy}}$; and supposing $\dot{y}$ invariable,

it will be $\dot{z}\dot{x}\sqrt{xx+yy}+\frac{\dot{z}xx\dot{x}}{\sqrt{xx+yy}}-\dot{z}\dot{x}\sqrt{xx+yy}$

the whole divided by $\dot{x}\dot{x}$, and this will be

$\frac{\dot{z}xx\dot{x}+\dot{z}xx\dot{y}-\dot{z}\dot{y}\dot{x}}{\dot{x}\dot{x}\sqrt{xx+yy}}$.

Taking x as constant, the Fluxion of

$\frac{y\dot{y}}{\sqrt{xx+yy}}$ will be $\dot{y}\dot{y}+y\ddot{y}\sqrt{xx+yy}-\frac{y\dot{y}\dot{y}}{\sqrt{xx+yy}}$,

which divided by $xx+yy$, and then it will be

$\frac{\dot{x}\dot{y}\dot{y}+\dot{y}\dot{y}\dot{y}+y\dot{x}\dot{y}}{xx+yy\sqrt{xx+yy}}$; and supposing $\dot{y}$ to be in-

variable, it will be $\frac{\dot{x}\dot{x}\dot{y}+\dot{y}\dot{y}\dot{y}-y\dot{y}\dot{x}}{xx+yy\sqrt{xx+yy}}$.

In like manner, the Fluxion of

$\frac{\dot{x}\dot{x}+\dot{y}\dot{y}\sqrt{xx+yy}}{-\dot{x}\dot{y}}$, or $\frac{\dot{x}\dot{x}+\dot{y}\dot{y}^{\frac{3}{2}}}{-\dot{x}\dot{y}}$, supposing x in-

variable, will be $-\frac{3\dot{x}\dot{y}\dot{y}\dot{x}\dot{x}+\dot{y}\dot{y}^{\frac{3}{2}}+\dot{x}\dot{y}\dot{x}\dot{x}+\dot{y}\dot{y}^{\frac{3}{2}}}{\dot{x}\dot{x}\dot{y}\dot{y}}$.

Here we must observe, that in the last Example, $\dot{y}$ cannot be supposed invariable; for if it be, its Fluxion $\ddot{y}$ will be nothing; and consequently must not come into the Quantity proposed.

DEFINITION II.

WHEN a Curve AFK is partly convex, FIG. 52,
 and partly concave, to a right Line AB 53, 54, 55.
 or Point B ; the Point F dividing the convex
 from the concave Part, or the End of the one,
 and the Beginning of the other, is called the
 Point of *Inflexion*, when the Curve being come
 to F , continues its Course towards the same
 Parts; and the Point of *Retrogression*, when the
 Curve returns back again towards the Place
 of its beginning.

PROP. II.

A General PROBLEM.

66. **T**HE Nature of the Curve AFK being
 given, to determine the Point of *Inflexion*
 or *Retrogression* F .

First, Let us suppose the Curve AFK to FIG. 52,
 have the right Line AB as a Diameter, and 53.
 the Ordinates PM , EF , &c. parallel to each
 other. Now if you draw the Ordinate FE
 from the Point F , and the Tangent FL ; and
 another Ordinate MP from the Point M in
 the concave Part AF of the Curve, as like-
 wise a Tangent MT . Then it is evident,

1^o, In Curves that have a Point of Inflexion,
 that while the Absciss AP constantly increa-
 ses, the Part AT of the Diameter, intercept-
 ed between the Vertex of the Diameter A and
 the Point T , where the Tangent meets the
 Diameter, does likewise increase until the Point
 P falls in E , after which it continually decrea-
 ses:

ses: therefore AT must become a *maximum* AL , when the Point P falls in E the Point sought.

2^o, That in those Curves that have a Point of Retrogression, the Part AT continually increases, and the Absciss AP , till the Point T falls or coincides with L ; and afterwards it constantly decreases; whence AE must be a *maximum*, when the Point T coincides with L .

Now call AE, x ; EF, y ; then will $AL = \frac{y\dot{x}}{\dot{y}} - x$, and the Fluxion of this will be $\frac{\dot{y}\dot{y}\dot{x} - y\dot{x}\ddot{y}}{\dot{y}\dot{y}} - \dot{x}$, supposing $\dot{x}$ invariable: which

(being divided by $\dot{x}$ the Fluxion of AE) must

* *Art. 47.* be $= 0$ *, or infinite: whence $-\frac{y\ddot{y}}{\dot{y}\dot{y}}$ is $= 0$

or Infinity: and so multiplying by $\dot{y}\dot{y}$, and dividing by $-\dot{y}$, there comes out $\dot{y} = 0$, or infinite. Now with this last Expression, and the general Equation of a Curve, the Point of Inflexion or Retrogression F may be found. For the Nature of the Curve AFK being given, we shall have a Value of $\dot{y}$ in $\dot{x}$, and throwing that Value into Fluxions, supposing $\dot{x}$ invariable, we shall get a Value of $\ddot{y}$ in $\dot{x}\dot{x}$, which being made equal to 0, and afterwards to Infinity: by means thereof, in either of these Suppositions, we may find AE so expressed, that its correspondent Ordinate EF shall intersect the Curve in the Point of Inflexion or Retrogression F .

The Point A whereat the x 's begin, may be so situate that $AL = x - \frac{y\dot{x}}{\dot{y}}$, instead of $\frac{y\dot{x}}{\dot{y}} - x$, and AL or AE a *minimum* instead of a *maximum*. But as the Confe-

Consequence is always the same, and there is no Difficulty arising from it, I shall say no more thereof.

Here we must observe that AL can never be $= x + \frac{y\dot{x}}{\dot{y}}$, for when the Point T falls on the other Side the Point P , with respect to A , the Value of $\frac{y\dot{x}}{\dot{y}}$ will be negative, (by *Art.* 10.) and consequently the Value of $-\frac{y\dot{x}}{\dot{y}}$ will be positive; so that in this

Case we have still $AE + EL$ or $AL = x - \frac{y\dot{x}}{\dot{y}}$.

The same things may be found after a different way. It is plain, taking x as invariable, *FIG.* 48, and supposing the Ordinate y to increase, that 49. Sn is less than SH or Rm in the concave Part of the Curve, and greater in the convex Part: therefore Hn ($\dot{y}\dot{y}$) which is positive, must become negative in the Point of Inflexion or Retrogression F ; and consequently * in that * *Art.* 47. Point it must be nothing or infinite.

Secondly, Let the Ordinates BM, BF, BM *FIG.* 54, of the Curve AFK all meet in the same Point 55.

B. If any Ordinate BM be drawn, and the Tangent MT meeting BT perpendicular to *FIG.* 56, BM in the Point T ; and if m be taken infi- 57.

nitely near M , and the Ordinate Bm , Tangent mt , the Perpendicular Bt to Bm meeting MT in O , be all drawn: then supposing the Ordinate BM , becoming Bm to increase, it is plain, in the concave Part of the Curve, that Bt is greater than BO , and in the convex Part the same is less. So that under the Point of Inflexion or Retrogression F , Ot must change from negative to positive.

G

This

FIG. 56. This being premised, about the Centre B describe the small Arches MR, TH , and there will be form'd the similar Triangles mRM, MBT, THO , and the little similar Sectors BMR, BTH . Now if you make $BM = y$; $MR = \dot{x}$. then will $mR(\dot{y}) : RM(\dot{x}) :: BM(y) : BT = \frac{y\dot{x}}{\dot{y}} :: MR(\dot{x}) : TH = \frac{\dot{x}\dot{x}}{\dot{y}} :: TH\left(\frac{\dot{x}\dot{x}}{\dot{y}}\right) : HO = \frac{\dot{x}\dot{x}\dot{x}}{\dot{y}\dot{y}}$. Now if you throw $BT\left(\frac{y\dot{x}}{\dot{y}}\right)$ into Fluxions, supposing $\dot{x}$ invariable, there comes out $Bt - BT$ or $Ht = \frac{\dot{x}\dot{y}\dot{y} - y\dot{x}\ddot{y}}{\dot{y}^2}$; and therefore $OH + Ht$ or $Ot = \frac{\dot{x}^2 + x\dot{y}^2 - y\dot{x}\ddot{y}}{\dot{y}^2}$.

Whence multiplying by $\dot{y}^2$, and dividing by $\dot{x}$, the Expression $\dot{x}^2 + \dot{y}^2 - y\ddot{y}$ must be nothing or infinite in the Point of Inflexion or Retrogression F . Now when the Nature of the Curve (*Fig. 54, 55.*) AFK is given, we shall have Values of $\dot{y}$ in $\dot{x}$, and of $\ddot{y}$ in $\dot{x}^2$, which being substituted in $\dot{x}^2 + \dot{y}^2 - y\ddot{y}$, there will arise a Quantity, which being made equal first to 0, and afterwards to Infinity; by means thereof we may get such a Value of BF , that if a Circle be described from B with the same as a Radius, it shall cut the Curve AFK in the Point of Inflexion or Retrogression F .

FIG. 50, 51. *Lastly*, To find the same things moreover another way, you must consider that the Angle BmE is greater than the Angle Bmn in the concave Part of the Curve; and the same is less in the convex Part: and therefore the

FIG. 50. Angle $BmE - Bmn$ or Emn , that is, the Arch En , being the Measure of it, changes from

from positive to negative in the sought Point *F*. Now taking $\dot{x}$ as invariable, because of the right-angled similar Triangles *HmS*, *Hnk*, we have $Hm(\dot{u}) : mS(\dot{x}) :: Hn(-\ddot{y}) : nk = -\frac{\dot{x}\ddot{y}}{\dot{u}}$. Where you must observe, that *Hn* is

negative, because as *Bm(y)* increases, *Rm(y)* decreases. But because of the similar Sectors *BmS*, *mEk*, therefore will $Bm(y) : mS(\dot{x}) :: mE(\dot{u}) : Ek = \frac{\dot{x}\dot{u}}{y}$; and so $Ek + kn$ or $En = \frac{\dot{x}\dot{u}^2 - y\dot{x}\ddot{y}}{y\dot{u}}$. Whence multiplying by $y\dot{u}$,

and dividing by $\dot{x}$, it follows that $\dot{u}^2 - y\ddot{y}$ or $\dot{x}^2 + \dot{y}^2 - y\ddot{y}$ must change from positive to negative under the sought Point *F*. FIG. 54.
55.

If y be supposed to become infinite, the Terms $\dot{x}^2$ and $\dot{y}^2$ will be nothing with respect to $y\ddot{y}$; and consequently the Form $\dot{x}^2 + \dot{y}^2 - y\ddot{y} = 0$, or infinite, will be changed into this $-y\ddot{y} = 0$, or infinite; that is, dividing by $-y$, $\ddot{y}$ will be $= 0$, or infinite, which is the Form of the first Case; and must be so likewise, because the Ordinates *BM*, *BF*, *EM* do then become parallel.

C O R O L.

67. **W**HEN $\ddot{y} = 0$, it is plain that the Fluxion of *AL* must be nothing with regard to the Fluxion of *AE*; and therefore the two infinitely near Tangents *FL*, *fL*, will coincide and make but one right Line *fFL*. But when $\ddot{y} = \text{Infinity}$, the Fluxion of *AL* must be infinitely great in respect to that of *AE*, or (which is the same thing) the Fluxion of *AE* is infinitely little with respect to FIG. 52.
FIG. 53.

the Fluxion of AL ; and consequently two Tangents FL , Fl may be drawn from the Point F , making an infinitely small Angle LFL .

FIG. 56,
57.

In like manner, when $\dot{x}^2 + \dot{y}^2 - y\ddot{y} = 0$, it is plain that Ot must become nothing in regard to MR ; and so the two infinitely near Tangents MT , mt , must coincide, when the Point M becomes a Point of contrary Flexion or Retrogression; but on the contrary, when $\dot{x}^2 + \dot{y}^2 - y\ddot{y} = \text{Infinity}$, Ot must be infinite with respect to MR , or (which is the same thing) MR infinitely small with regard to Ot : and consequently the Point m must coincide with M , that is, one may draw two Tangents thro' the Point M , making an infinitely small Angle with each other, when that Point becomes a Point of Flexion, and contrary Retrogression.

Hence it follows, that the Tangent to the Point of Inflexion or Retrogression F , continued out, does both touch and cut the Curve AFK in that Point.

EXAMPLE I.

FIG. 58. 68. **L**ET AFK be a Curve, the right Line AB being a Diameter of it, and let the Relation of any Absciss AE (x) to its correspondent Ordinate EF (y) be expressed by this Equation $axx = xxy + aay$. It is required to find such an Expression for AE , that the Ordinate EF shall intersect the Curve AFK in the Point of Inflexion F .

The Equation of the Curve is $y = \frac{axx}{xx + aa}$; and

and therefore $y = \frac{2a^3 x \dot{x}}{xx + aa}$, and finding the Fluxion of this Quantity, supposing $\dot{x}$, to be invariable, and making it equal to nothing, there will arise $\frac{2a^3 \dot{x}^2 \times xx + aa^2 - 3a^3 xx \dot{x}^2 \times xx + aa}{xx + aa^4} = 0$.

This multiply'd by $xx + aa^4$, and afterwards divided by $2a^3 \dot{x}^2 \times xx + aa^2$, will be $xx + aa - 4xx = 0$. Whence $AE(x) = \frac{a}{\sqrt{3}}$.

If $\frac{1}{3}aa$ be put for its Equal xx in the Equation of the Curve $y = \frac{axx}{xx + aa}$, we shall have $EF(y) = \frac{1}{4}a$; so that the Point of Inflexion F may be determined without supposing the Curve AFK to be described.

If AC be drawn parallel to the Ordinates EF , and made equal to the given right Line a , and CG be drawn parallel to AB , the same will be an Asymptote to the Curve AFK . For if x be supposed infinite, we may take xx for $xx + aa$; and therefore the Equation of the Curve $y = \frac{axx}{xx + aa}$ will be changed into this $y = a$.

EXAMPLE II.

69. **L**ET $y - a = x - a^{\frac{2}{3}}$. Then $\dot{y} = \frac{1}{3}x - a^{\frac{2}{3}}\dot{x}$, and $\ddot{y} = -\frac{6}{25}x - a^{\frac{7}{5}}\dot{x}^2 = \frac{-6\dot{x}^2}{25\sqrt[5]{x - a}}$ supposing $\dot{x}$ invariable. Now if the latter Expression be made equal to 0, there will come out $-6\dot{x}^2 = 0$. Which determines nothing; therefore this last Expression must be supposed infinitely great; and consequently the Deno-

minator $2\sqrt{x-a}$ thereof infinitely small, or $=0$. Whence the unknown Quantity $AE(x) = a$.

EXAMPLE III.

FIG. 59. 70. **L**ET AFK be a Semi-cycloid, the Base BK of which let be greater than the Semi-circumference ADB of the generating Circle, whose Centre is C . It is required to find the Point E in the Diameter AB , from which the Ordinate EF being drawn, will cut the Cycloid in the Point of Inflexion F .

Call the known Quantities ADB, a ; BK, b ; $AB, 2c$; and the unknown ones AE, x ; ED, z ; the Arch AD, u ; EF, y . Then from the Nature of the Cycloid $y = z + \frac{bu}{a}$: and there-

fore $y = z + \frac{bu}{a}$. Now by the Nature of the

Circle we have $z = \sqrt{2cx - xx}$, $\dot{z} = \frac{c\dot{x} - x\dot{x}}{\sqrt{2cx - xx}}$,

and $\dot{u}(\sqrt{\dot{x}^2 + \dot{z}^2}) = \frac{c\dot{x}}{\sqrt{2cx - xx}}$. Whence sub-

stituting for $\dot{z}$ and $\dot{u}$ their Equals, we have

$\dot{y} = \frac{ac\dot{x} - ax\dot{x} + bc\dot{x}}{a\sqrt{2cx - xx}}$. The Fluxion of

which (supposing x invariable) will be $\frac{bcx - acc - bccx^2}{2cx - xx \times \sqrt{2cx - xx}} = 0$. From whence ari-

ses $AE(x) = c + \frac{ac}{b}$, and $CE = \frac{ac}{b}$.

It is manifest, if there be a Point of Inflexion F , that C must be greater than a ; for if it be less, CE will be greater than CB .

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EXAMPLE IV.

71. **L**ET *AFK* be the Conchoid of *Nico-medes*, and the Point *P* the Pole, and the right Line *BC* the Asymptote. It is required to find the Point of Inflection *F*. Now the Nature of this Curve is such, that a right Line *PF* drawn from any Point *F* in it to the Pole *P* meeting the Asymptote *BC* in *D*; the Part *DF* thereof is always equal to a standing Line *a*.

Draw *PA* perpendicular, and *EF* parallel to *BC*, and call the known Quantities *AB* or *FD* *a*; *BP*, *b*; and the unknown ones *BE*, *x*; *EF*, *y*; and draw *DL* parallel to *BA*. Then because of the similar Triangles *DLF*, *PEF*, $DL(x) : LF(\sqrt{aa-xx}) :: PE(b+x) : EF(y)$
 $= \frac{b+x\sqrt{aa-xx}}{x}$. The Fluxion of this will be

$$\dot{y} = \frac{x^3 \dot{x} + aab\dot{x}}{xx\sqrt{aa-xx}}$$

Now if you take the Fluxion of it again, and make the same equal to nothing,

$$\text{there will arise } \frac{2a^4b - aax^3 - 3aabxx\dot{x}^2}{aax^3 - x^3x\sqrt{aa-xx}} = 0,$$

which may be brought down to $x^3 + 3bxx - 2aab = 0$; and one of the Roots thereof will be the Quantity of *BE* sought.

If $a=b$, the Equation aforegoing will become $x^3 + 3axx - 2a^3 = 0$; which divided by $x+a$, and there comes out $xx + 2ax - 2aa = 0$; and so $BE(x) = -a + \sqrt{3aa}$.

Otherwise.

Conceive the Lines *PF* issuing from the Pole *P* as Ordinates, and use the Form $*y\dot{y} = \dot{x}^2$ * *Art. 66.* $+ \dot{y}^2$, where $\dot{x}$ is supposed invariable. Now if

you conceive another Ordinate Pf making an infinitely small Angle FPf with PF , and describe the small Arches FG, DH , from the Centre P . Call the known Quantities AB, a ; BP, b ; and the unknown ones PF, y ; PD, z . Then from the Nature of the Conchoid $y = z + a$, and so $\dot{y} = \dot{z}$. Now from the right-angled Triangle DBP , $DB = \sqrt{zz - bb}$, and because of the similar Triangles $DBP, dHD; PDH, PFG$, $DB(\sqrt{zz - bb}) : BP(b) :: dH(\dot{z}) : HD$

$$= \frac{b\dot{z}}{\sqrt{zz - bb}}$$
; and $PD(z) : PF(z + a) :: HD$

$$\frac{b\dot{z}}{\sqrt{zz - bb}} : FG(\dot{x}) = \frac{bz\dot{z} + ab\dot{z}}{z\sqrt{zz - bb}}. \text{ Whence we}$$

get $\dot{z}$ or $\dot{y} = \frac{z\dot{x}\sqrt{zz - bb}}{bz + ab}$, the Fluxion of

$$\text{which, supposing } \dot{x} \text{ invariable, will be } \ddot{y} = \frac{bz^3 + 2abzz - ab^3 \times \dot{z}\dot{x}}{bz + ab \times \sqrt{zz - bb}} = \frac{bz^4 + 2abz^2 - ab^3 \times \dot{z}\dot{x}^2}{bz + ab^3},$$

substituting for $\dot{z}$ what is equal to it. Now
 ¶ Art. 66. if in the general Form $*y\dot{y} = \dot{x}^3 + \dot{y}^2$, you put $z + a$ for y , and for $\dot{y}$ and $\ddot{y}$, the Values already found in $\dot{x}$ and $\dot{x}^2$, we shall have

$$\frac{z^4 + 2az^2 - abbz \times \dot{x}^2}{bz + ab^3} = \frac{z^4 + 2abbz + aabb \times \dot{x}^2}{bz + ab^3},$$

which may be brought down to $2z^3 - 3bbz - abb = 0$; and one of the Roots of this Plus a , will express PF sought.

If $a = b$, then will $2z^3 - 3aaz - a^3 = 0$, which divided by $z + a$, will be $zz - az - \frac{aa}{2} = 0$;

$$\text{whence } PF(z + a) = \frac{1}{2}a + \frac{1}{2}a\sqrt{3} = \frac{3a + a\sqrt{3}}{2}.$$

EXAMPLE

EXAMPLE V.

72. **L**ET *AFK* be a Conchoid of another Kind, the Nature whereof is such, that if a right Line *PF* be drawn from any Point *F* taken in it to the Pole *P*, intersecting the Asymptote *BC* in *D*, the Rectangle *PD* $\times$ *DF*, will be always = to a constant Rectangle *PB* $\times$ *BA*. It is required to find the Point of Inflexion *F*. FIG. 60.

Call the unknown Quantities *BE*, *x*; *EF*, *y*; and the known ones *AB*, *a*; *BP*, *b*; then will *PD* $\times$ *DF* = *ab*: And because of the Parallels *BD*, *EF*, *PD* $\times$ *DF* (*ab*): *PB* $\times$ *BE* (*bx*):: *PF*² (*bb* + *2bx* + *xx* + *yy*): *PE*² (*bb* + *2bx* + *xx*).

Whence *bbx* + *2bxx* + *x³* + *yyx* = *abb* + *2abx* + *axx*, or *yy* = $\frac{abb + 2abx + axx - bbx - 2bxx - x^3}{x}$,

and *y* = $\frac{b + x\sqrt{a-x}}{x} = \sqrt{ax-xx} + b\sqrt{\frac{a-x}{x}}$, the

Fluxion of which will be $\dot{y} = \frac{ax\dot{x} + 2xx\dot{x} + ab\dot{x}}{2x\sqrt{ax-xx}}$.

And again, throwing this into Fluxions, there will arise $\frac{3aab - aax - 4abx \times \dot{x}^2}{4axx - 4x^3 \times \sqrt{ax-xx}} = 0$, which

may be brought down to $x = \frac{3ab}{a+4b} = BE$.

If you make $\frac{-ax\dot{x} + 2xx\dot{x} + ab\dot{x}}{2x\sqrt{ax-xx}}$ the Value of

$\dot{y} = 0$; then will $xx - \frac{1}{2}ax + \frac{1}{2}ab = 0$, the two

Roots $\frac{a + \sqrt{aa - 8ab}}{4}$ and $\frac{a - \sqrt{aa - 8ab}}{4}$

whereof, will be such Expressions of *BH* and *BL*, when *a* exceeds *8b*, that the Ordinate FIG. 61.

HM

HM shall be less than those near it, and the Ordinate LN greater, *viz.* the Tangents to the Curve in M and N , will be parallel to the Axis AB ; and then the Point E will fall between H and L .

FIG. 62. But when $a=8b$, the Lines BH , BE , BL , will be equal each to $\frac{1}{2}a$. And then the Tangent in the Point of Inflexion F , will be parallel to the Axis AB . Lastly, when a is less than $8b$, the two Roots are imaginary, and so no Tangent can be parallel to the Axis.

FIG. 60. This Problem may be solved likewise in taking the Lines PF , Pf , issuing from the Pole P as Ordinates, and using the Form $y\ddot{y} = \dot{x}^2 + \dot{y}^2$, as in the last Example.

EXAMPLE VI.

FIG. 63. 73. **L**ET AED be a Circle, whose Centre is the Point B ; and let AFK be a Curve of such a Nature, that any Radius BFE being drawn, the Square of FE be equal to the Rectangle under the Arch AE , and a given right Line b . It is required to find the Point of Inflexion F of that Curve.

Make the Arch $AE=z$, the Radius BA or $BE=a$; and the Ordinate $BF=y$; then will $bz=aa-2ay+yy$, and (throwing it into Fluxions) $\frac{2y\dot{y}-2a\dot{y}}{b}=\dot{z}=Ee$. Now because of the similar Sectors BEe , BFG , $BE(a):BF(y)::Ee\left(\frac{2y\dot{y}-2a\dot{y}}{b}\right):FG(\dot{x})=\frac{2yy\dot{y}-2ay\dot{y}}{ab}$. The Fluxion of this, supposing $\dot{x}$ invariable, will be $4y\dot{y}^2-2a\dot{y}^2+2yy\ddot{y}-2ay\ddot{y}=0$; and therefore $y\ddot{y}=\frac{a\dot{y}^2-2y\dot{y}^2}{y-a}$. If now for $\dot{x}^2$ and $y\ddot{y}$,

you

you put their Equals in $\dot{y}^2$, in the general Form * $y\ddot{y} = \dot{x}^2 + \dot{y}^2$, there will arise $\frac{a\dot{y}^2 - 2y\dot{y}^2}{y - a}$ * Art. 66.

$$= \frac{4y^4\dot{y}^2 - 8ay^3\dot{y}^2 + 4aayyy\dot{y}^2 + aabb\dot{y}^2}{aabb}; \text{ which}$$

may be brought to $4y^5 - 12ay^4 + 12aay^3 - 4a^3yy + 3aabby - 2a^3bb = 0$; and so one of the Roots of this will express BF sought.

It is manifest, that the Curve AFK , which may be called a *Parabolical Spiral*, must have a Point of Inflexion F : For because the Circumference AED does not at first sensibly differ from the Tangent in A , it follows, from the Nature of the Parabola, that it must at first be concave towards that Tangent; and afterwards, when the Curvature of the Circumference becomes sensible about that Centre, it must become concave towards the said Centre.

EXAMPLE VII.

74. **L**ET AFK be a Curve, whose Axis is the right Line AB ; and let the Nature of it be such, that any Tangent FB being drawn meeting AB in the Point B , the intercepted Part AB will be always to the Tangent BF , in the given Ratio of m to n . It is required to determine the Point of Retrogression F . FIG. 64.

Call the variable Quantities AE, x ; EF, y ; then $EB = -\frac{y\dot{x}}{\dot{y}}$ (because when x increases, y decreases) and $FB = \frac{y\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{y}}$. Now from the Nature of the Curve, $AE + EB$ or AB $\left(\frac{x\dot{y} - y\dot{x}}{\dot{y}}\right) : BF \left(\frac{y\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{y}}\right) :: m : n$. There-

fore

fore $m\sqrt{x^2+y^2} = \frac{nx\dot{y}}{y} - n\dot{x}$; and the Fluxion will be $\frac{m\ddot{y}\dot{y}}{\sqrt{x^2+y^2}} = \frac{-ny\dot{x}\dot{y} + nxy\ddot{y} - nxy^2}{yy}$, sup-
 posing $\dot{x}$ invariable and negative; whence we get $\ddot{y} = \frac{ny\dot{x}\dot{y} - nxy^2\sqrt{x^2+y^2}}{m\dot{y}\dot{y} - nxy\sqrt{x^2+y^2}}$. Now making
 this last fractional Expression equal to nothing, and there will be had $-y\dot{x} - x\dot{y} = 0$, from which nothing can be determined. But if the said fractional Expression be made infinite, viz. the Denominator equal to 0, then we shall have $\sqrt{x^2+y^2} = \frac{m\dot{y}\dot{y}}{nx} = \frac{nx\dot{y} - ny\dot{x}}{m\dot{y}}$ because of the E-
 quation of the Curve; and consequently $\dot{x} = \frac{nnxx\dot{y} - mm\dot{y}\dot{y}\dot{y}}{mm\dot{y}\dot{y}}$. Now squaring both Sides of the Equation $m\dot{y} = nx\sqrt{x^2+y^2}$, and we shall have likewise $\dot{x} = \frac{\dot{y}\sqrt{mm\dot{y}\dot{y} - nnxx}}{nx}$
 $\frac{nnxx\dot{y} - mm\dot{y}\dot{y}\dot{y}}{nnxy}$. From whence, at length,

there comes out $y\sqrt{mm - nn} = nx$; and so we get the following Construction.

On the Diameter $AD = m$, describe a Semicircle AID , and take the Cord $DI = n$, and draw the Line AI . This will intersect the Curve AFK in the Point of Retrogression F .

For if IH be drawn perpendicular to AB , because of the similar right-angled Triangles $DIA, IHA, FEA, DI(n):IA(\sqrt{mm - nn}) :: IH:HA :: FE(y):EA(x)$. And therefore $y\sqrt{mm - nn} = nx$.

It is manifest that BF is parallel to DI ; because $AB:BF :: AD(m):DI(n)$. Whence the

the Angle AFB is a right Angle, and so the Lines AB, BF, BE , are continued Proportionals.

This Property will appear from other Principals thus. Conceive * FB, Fb , two Tan- * *Art. 67.*
gents infinitely near each other in the Point F of Retrogression, making the infinitely small Angle $B F b$. Then if the small Arch BL be described from the Centre F , there will be had
 $m : n :: Ab : bF :: AB : BF :: Ab - AB$ or $Bb : bF - BF$ or $bL :: BF : BE$. Because of the right-angled similar Triangles BbL, FBE ; whence, &c.

If $m = n$, it is plain that the right Line AF will become perpendicular to the Axis AB ; and so the Tangent FB will be parallel to the same; which that it is so, is otherwise evident, because in this Case the Curve AF must be a Semicircle, a Diameter whereof is perpendicular to the Axis AB . But if m be less than n , the Curve cannot have a Point of Retrogression, because then $y\sqrt{mm - nn} = nx$ is an impossible Equation.





S E C T. V.

Use of Fluxions in the Doctrine of Evolute and Involute Curves.

D E F I N I T I O N.

FIG. 65. **I**F one End F of the Thread $ABDF$ be first fixed to the Point F in a Curve BDF concave the same way; and afterwards the Thread be put about the said Curve, so as to touch it in every Part. Then if the other End A of the Thread be tightly moved in the same Plane with the Curve, so as to continually disengage itself from the Curve BDF , the said End A of the Thread, will describe a Curve AHK , which is called an *Involute Curve* or *Figure*.

And the Curve BDF is called the *Evolute* of the Curve AHK , or the *Evolute Curve*.

And the straight Parts AB , HD , KF , of the Thread, are each called the *Radius* of *Evolution*, or of the *Evolute*.

C O R O L. I.

75. **B**ECAUSE the Length of the Thread $ABDF$ does not vary, it is plain that the Length of the Part BD of the Curve, is equal to the Difference between the Radii HD and AB issuing from the Ends thereof. In like

like manner, the Length of the Part DF , will be equal to the Difference between the Radii FK, DH ; and the Length of the whole Curve BDF , to the Difference of the Radii FK, BA . Whence if the Radius BA of the Curve be o , or the End A of the Thread falls in B , the Origin of the Curve BDF ; then will the Radii of the Evolute DH, FK , be equal to the Lengths of the Parts BD, BDF , of the Curve BDF .

COROL. II.

76. **I**F the Curve BDF be conceived as a Polygon $BCDEF$ of an infinite Number of Sides, it is evident, that A the End of the Thread $ABCDEF$ will describe the small Arc AG , whose Centre is the Point C , until the said Radius CG coincides with the Continuation of the little Side CD adjoining to CB ; and then it will describe the small Arch GH , having the Point D as a Centre, until DH coincides with the Continuation of the little Side DE ; and so on, until the Thread be quite disengaged from the Curve $BCDEF$. Therefore the Curve AHK must be consider'd as the Assemblage of an infinite Number of small circular Arches $AG, GH, HI, IK, \&c.$ having the Points $C, D, E, F, \&c.$ as Centres. Whence,

1^o, The Radii of Evolution touch the Curve continually, viz. DH in D, KF in $K, \&c.$ For DH , for Example, is perpendicular to the small Arches GH and HI , because it passes thro' their Centres D, E . Whence we may observe, 1^o, That the evolute Curve BDF , terminates the Space wherein all the Perpendiculars

FIG. 66.

FIG. 65.

diculars to the Curve AHK fall. 2^o, That if any Radius HD be continued out, intersecting the Radius AB in R , until it meets any other Radius KF in S . We can draw always from all the Points in the Part RS , two Perpendiculars to the Curve AHK , except from the Point of Contact D ; from whence but one Tangent DH can be drawn. For it is evident that R , the Intersection of the Radii AB , DH , runs thro' all the Points in the Part RS , while the End A , of the Radius AB , describes the Line AHK , to which it is continually perpendicular: And that the Radii AB , HD , do not coincide, but when the Intersection R falls in the Point of Contact D .

FIG. 66. 2^o, If the little Arches be continued out, viz. HG to l , IH to m , KI to n , &c. towards A . Every little Arch, as IH , will outwardly touch HG the little Arch next to it, because the Radii CA , DG , EH , FI , increase so much the more as the small Arches, the Curve AHK consists of, are farther from the Point A . In like manner, if you continue out the little Arches AG to o , GH to p , HI to q , towards contrary Parts from the Point A ; every little Arch, as HI , will touch inwardly the small Arch IK next to it. Now because the Points H and I , D and E , on account of the infinitely Smallness of the Arch HI and the Side DE , may be consider'd as coinciding: Therefore, if from any intermediate Point D of the evolute BDF , as a Centre, with the Radius DH , you describe a Circle mHp , it will outwardly touch the Part HA , which will entirely fall within that Circle, and inwardly the other Part HK , which will fall quite without the said Circle: That

is,

is, it will both touch and cut the Curve AHK in the Point H ; just as the Tangent in the Point of Inflection does cut the Curve in that Point.

3°, Because the Radius HD of the little Arch HG , differs from the Radii CG , EH , of the Arches GA , HI , next to it, only by the infinitely small Quantity CD or DE ; therefore if the Radius HD be lessen'd never so little, it shall be less than CG ; and so its Circle will inwardly touch the Part HA ; and, contrariwise, if it be never so little increased, it will be greater than HE , and so the Circle thereof will touch the Part HK outwardly. Whence the Circle mHp is the least of all those that touch the Part HA outwardly, and the greatest of all those that touch the Part HK inwardly; that is, no Circle can be drawn between this and the Curve.

4°, Because the Curvature of Circles increases in the same Proportion as their Radii decrease; therefore the Curvature of the small Arch HI , will be to the Curvature of the small Arch AG reciprocally as the Radius BA or CA of this latter, to its Radius DH or EH : That is, the Curvature in H of the Curve AHK , will be to the Curvature in A , as the Radius BA to the Radius DH ; and in like manner, the Curvature in K is to the Curvature in H , as the Radius DH is to the Radius FK . Whence it is manifest, that the Curvature of the Line AHK , decreases so much the more as the Radius of the evolute Curve BDF is greater; so that in the Point A , the beginning of the Curve, it will be a *Maximum*, and at the Point K , where it is supposed to end, a *Minimum*.

H

5°, That

5°. That the Points of the evolute Curve, are only the Intersections of the Perpendiculars drawn from the Ends of little Arches, whereof the Curve AHK is supposed to consist. For Example, the Point D or E is the Intersection of the * Perpendiculars HD , IE , to the small Arch HI ; so that if the Curve AHK be given together with the Position of one of the Perpendiculars HD to it, and it be required to find the Point D or E , wherein the Perpendicular touches the evolute Curve; the Business is only to find the Point wherein the Perpendiculars HD , IE , infinitely near each other do meet: And this is the Work of the following general Problem.

P R O P. I.

FIG. 67. 77. **T**HE Nature of the Curve AMD being given, together with the Position of a Perpendicular MC to it; to find the Length of the Radius MC of the evolute Curve; or, which is the same thing, to find the Point wherein the Perpendiculars MC , mC , infinitely near each other, meet.

First, Let the Ordinates PM of the Curve AMD , be perpendicular to the Axis AB ; and let the Ordinate mp be infinitely near MP , because the Point m is supposed infinitely near M . From the Point of Intersection C , draw CE parallel to the Axis AB , meeting the Ordinates MP , mp , in the Points E , e . Lastly, draw MR parallel to AB , and then the right-angled Triangles MRm , MEC , will be similar: Since the Angles EMR , CMm , being right Angles, and the Angle CMR being common

* These Perpendiculars are called the Radii of the Curvature.

mon to them, the Angle EMC shall be equal to the Angle RMm .

Now call the given Quantities AP, x ; PM, y ; and the unknown one ME, z . Then will Ee , or Pp , or $MR = \dot{x}$, $Rm = \dot{y} = \dot{z}$, $Mm = \sqrt{\dot{x}^2 + \dot{y}^2}$; and $MR (\dot{x}) : Mm (\sqrt{\dot{x}^2 + \dot{y}^2}) :: ME (z) : MC = \frac{z\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}}$. Now since the

Point C is the Centre of the small Arch Mm , the Radius CM thereof, which becomes Cm while EM increased by its Fluxion Rm continues invariable; therefore the Fluxion of it will then be nothing; and so making $\dot{x}$ invariable, $\frac{\dot{z}\dot{x}\dot{x} + \dot{z}\dot{y}\dot{y} + z\dot{y}\dot{y}}{\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2} + \dot{y}\dot{y}} = 0$. Whence we get

$ME (z) = \frac{\dot{z}\dot{x}\dot{x} + \dot{z}\dot{y}\dot{y}}{-\dot{y}\dot{y}} = \frac{\dot{z}\dot{x} + \dot{y}\dot{y}}{-\dot{y}}$ by substituting $\dot{y}$ for $\dot{z}$.

2°, Let the Ordinates BM, Bm , all issue from the same Point B . Draw the Perpendiculars CE, Ce , from the sought Point C to the Ordinates, which suppose to be infinitely near each other, and describe the small Arch MR from the Centre B ; then the right-angled Triangles RMm and EMC, BMR, BEG and CeG will be similar. And making $BM = y$, $ME = z$, $MR = \dot{x}$, and we shall have $Rm = \dot{y}$, $Mm = \sqrt{\dot{x}^2 + \dot{y}^2}$, CE or $Ce = \frac{z\dot{y}}{\dot{x}}$, and $MC = \frac{z\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}}$.

And, as before, we shall get $z = \frac{\dot{z}\dot{x}^2 + \dot{z}\dot{y}^2}{-\dot{y}\dot{y}}$.

Now $BM (y) : Ce \left(\frac{z\dot{y}}{\dot{x}} \right) :: MR (\dot{x}) : Ge = \frac{z\dot{y}}{\dot{y}}$, and $me = ME$ or $Rm = Ge = \dot{z} = \frac{\dot{y}\dot{y} - z\dot{y}}{\dot{y}}$. Whence substituting this for $\dot{z}$,
H 2 and

and $ME (z)$ will be $= \frac{y\dot{x} + y\ddot{y}}{\dot{x}^2 + \dot{y}^2 - y\ddot{y}}$.

If y be supposed infinite, the Terms $\dot{x}^2$ and $\dot{y}^2$, will be nothing in respect of $y\ddot{y}$; and consequently this last Form will fall into that of the Case foregoing. This must be so, because the Ordinates then do become parallel, and the Arch MR a right Line perpendicular to the Ordinates.

Now if the Nature of the Curve AMD be given, we may get the Values of $\dot{y}^2$ and $\ddot{y}$ in $\dot{x}^2$, or of $\dot{x}^2$ and $\ddot{y}$ in $\dot{y}^2$; which being substituted in the precedent Forms, and there will arise an Expression for ME freed from Fluxions. And then drawing EC perpendicular to ME , it shall cut MC perpendicular to the Curve, in the sought Point C .

COROL. I.

FIG. 67, 78. **B**ECAUSE of the right-angled similar
68. Triangles MRm and MEC , in the

former Case $MC = \frac{\dot{x}^2 + \dot{y}^2 \sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\ddot{y}}$, and in

the latter $MC = \frac{y\dot{x}^2 + y\dot{y}^2 \sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}^3 + \dot{x}\dot{y}^2 - y\dot{x}\ddot{y}}$.

SCHOLIUM.

79. **T**HERE are other ways of finding the Radii of Evolution; some of which I shall mention for the sake of those who have not been acquainted with Investigations of this kind.

CASE I. In Curves whose Ordinates are perpendicular to the Axis.

1st Way.

1st Way. Continue out MR to G , wherein FIG. 67.
 it intersects the Perpendicular mC . Because
 the Angles MRm , MmG are right ones, RG
 will be $= \frac{y^2}{\dot{x}}$; and so $MG = \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}}$. Now
 since the Triangles MRm , MPQ (the Points
 Q, q being the Intersections of the infinitely
 near Perpendiculars, with the Axis AB) are si-
 milar, there arises $MQ = \frac{y\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}}$, $PQ =$
 $\frac{y\dot{y}}{\dot{x}}$; and therefore $AQ = x + \frac{y\dot{y}}{\dot{x}}$, the Flu-
 xion whereof (supposing $\dot{x}$ invariable) brings
 out $Qq = \dot{x} + \frac{\dot{y}^2 + y\ddot{y}}{\dot{x}}$; and because of the si-
 milar Triangles CMG , CQq , $MG - Qq$
 $\left(\frac{-y\ddot{y}}{\dot{x}}\right) : MG \left(\frac{\dot{x}^2 + \dot{y}^2}{\dot{x}}\right) :: MQ \left(\frac{y\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}}\right)$
 $: MC = \frac{\dot{x}^2 + \dot{y}^2 \sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\ddot{y}}$.

2d Way. From the Centre C , describe the
 small Arch QO ; then the little right-angled
 Triangles QOq , MRm , will be similar, be-
 cause Mm , QO and MR , Qq , are parallel.
 Therefore $Mm (\sqrt{\dot{x}^2 + \dot{y}^2}) : MR (\dot{x}) :: Qq$
 $\left(\frac{\dot{x}^2 + \dot{y}^2 + y\ddot{y}}{\dot{x}}\right) : QO = \frac{\dot{x}^2 + \dot{y}^2 + y\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$. Now
 because of the similar Sectors CMm , CQO ,
 $Mm - QO \left(\frac{-y\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) : Mm (\sqrt{\dot{x}^2 + \dot{y}^2}) ::$
 $MQ \left(\frac{y\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}}\right) : MC = \frac{\dot{x}^2 + \dot{y}^2 \sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\ddot{y}}$.

3d Way. If the Tangents MT , mt , be
 drawn infinitely near each other, then will

$PT \rightarrow AP$ or $AT = \frac{y \dot{x}}{y} - x$, the Fluxion where-
of will be $Tt = -\frac{y \ddot{x} \dot{y}}{y^2}$; and if the small Arch

TH be described from the Centre m , the right-angled Triangle HTt will be similar to the right-angled Triangle RmM ; for the Angles HTt , RmM or PTM are equal, since their Difference is the Angle Tmt , which is infinitely small. Whence $Mm (\sqrt{\dot{x}^2 + \dot{y}^2}) : mR$

$$(\dot{y}) :: Tt \left(\frac{-y \ddot{x} \dot{y}}{y^2} \right) : TH = \frac{-y \ddot{x} \dot{y}}{y \sqrt{\dot{x}^2 + \dot{y}^2}}. \text{ Now}$$

the Sectors TmH , MCm , are similar; for the Angles $TMt + MmC = \text{one right Angle}$, and the Angles $MmC + MCm$ are likewise equal to one right Angle, because the Triangle CMm is consider'd as right-angled at M . Therefore

$$TH \left(\frac{-y \ddot{x} \dot{y}}{y \sqrt{\dot{x}^2 + \dot{y}^2}} \right) : Mm (\sqrt{\dot{x}^2 + \dot{y}^2}) :: Tm \text{ or}$$

$$TM \left(\frac{y \sqrt{\dot{x}^2 + \dot{y}^2}}{y} \right) : MC = \frac{\dot{x}^2 + \dot{y}^2 \sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x} \ddot{y}}.$$

FIG. 69.

* Art. 64.

4th Way. Taking $\dot{x}$ as invariable, the second Fluxions must be denoted $\ddot{y}$; and because of the right-angled similar Triangles HmS , Hnk , Hm or $Mm (\sqrt{\dot{x}^2 + \dot{y}^2}) : mS$ or $MR (\dot{x})$
 $:: Hn (-\ddot{y}) : nk = -\frac{\dot{x} \ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}.$ Now the An-

gle kmn is equal to that made by the Tangent at the Points M, m , and therefore equal to the Angle MCm ; whence the Sectors mnk , MCm ,

are similar; and so $nk \left(\frac{-\dot{x} \ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) : mk$ or

* Art. 2. $Mm (\sqrt{\dot{x}^2 + \dot{y}^2}) : MC = \frac{\dot{x}^2 + \dot{y}^2 \sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x} \ddot{y}}.$ Note,

mH

mH or Mm is taken for mk , because their Difference is only the short Line Hk infinitely less than either of them. In like manner, as Hn is infinitely less than Rm or Sn .

CASE II. In Curves whose Ordinates issue all from a given Point.

1st Way. Draw the Perpendiculars BF, Bf , FIG. 68.
from the Point B to the infinitely-near Radii CM, Cm ; then because the right-angled Triangles mMR, BMF , are similar (since if the same Angle FMR be added to the Angles mMR, BMF , each of the Sums will be equal

to a right Angle) MF or $MH = \frac{y\dot{x}}{\sqrt{x^2 + y^2}}$,
and $BF = \frac{y\dot{y}}{\sqrt{x^2 + y^2}}$, the Fluxion whereof is
 $Bf - BF$, or $Hf = \frac{\dot{x}^2 y^2 + \dot{y}^4 + y\dot{x}^2 \ddot{y}}{\dot{x}^2 + \dot{y}^2 \times \sqrt{x^2 + y^2}}$. Suppo-

sing $\dot{x}$ invariable. Now since the Sectors CMm, CHf , are similar, therefore $Mm - Hf : Mm$
:: $MH : MC$; and so $MC = \frac{y\dot{x}^2 + y\dot{y}^2 \sqrt{x^2 + y^2}}{\dot{x}^2 + \dot{x}\dot{y}^2 - y\dot{x}\ddot{y}}$.

2d Way. Denote * the second Fluxions in * Art. 64.
making $\dot{x}$ invariable; then because of the si- FIG. 70.
milar Sectors $BmS, mEk, BM(y) : mS(\dot{x}) ::$

$mE(\sqrt{\dot{x}^2 + \dot{y}^2}) : Ek = \frac{\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}}{y}$. Now because
of the right-angled similar Triangles $HmS,$
 Hnk, Hm or $Mm(\sqrt{\dot{x}^2 + \dot{y}^2}) : mS$ or $MR(\dot{x})$
:: $Hn(-\ddot{y}) : nk = -\frac{\dot{x}\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$. And there-

fore $En = \frac{\dot{x}^2 + \dot{x}\dot{y}^2 - y\dot{x}\ddot{y}}{y\sqrt{\dot{x}^2 + \dot{y}^2}}$; and finding a third
Proportional to En, Em or Mm , by means of
H 4 the

the similar Sectors Emn , MCm , the same Expression for MC , as before, will be had.

If you make $Mm(\sqrt{\dot{x}^2 + \dot{y}^2}) = \dot{u}$; and y be taken instead of $\dot{x}$, as invariable. In the former Case, $MC = \frac{\dot{u}^3}{y\ddot{x}}$, and in the other $MC =$

$\frac{y\dot{u}^3}{\dot{x}\dot{u}^3 + y\dot{y}\ddot{x}}$. And lastly, if $\dot{u}$ be supposed invariable, there comes out in the former Case

$MC = \frac{\dot{x}\dot{u}}{-\ddot{y}}$ or $\frac{y\dot{u}}{\ddot{x}}$ (because the Fluxion of $\dot{x}^2 + \dot{y}^2 = \dot{u}^2$ is $\dot{x}\ddot{x} + \dot{y}\ddot{y} = 0$; and so $\frac{\dot{x}}{-\ddot{y}} = \frac{y}{\ddot{x}}$)

and in the latter, $MC = \frac{y\dot{x}\dot{u}}{\dot{x}^2 - y\ddot{y}}$ or $\frac{y\dot{u}}{\dot{x}\dot{y} + y\ddot{x}}$.

COROL. II.

FIG. 72. 80. **S**INCE ME or MC has been found to have but one Value, therefore the involute Curve AMD has but one evolute BCG .

COROL. III.

FIG. 67, 81. 68. **I**F $ME \left(\frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}} \right)$ or $\left(\frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 + \dot{y}^2 - y\ddot{y}} \right)$

be positive, the Point E must be taken on the same Side the Axis AB or Point B , as it was supposed in the Operation foregoing. And so the Curve in that Case will be concave towards the Axis, or that Point. But if ME be negative, the Point E must be taken on the contrary Side, and so the Curve will be then convex towards the Axis, or that Point. Therefore it is plain, that in the Point of Inflexion
or

or Retrogression, which separates the concave from the convex Part, ME from being positive will become negative. Whence the contiguous or infinitely-near Perpendiculars from converging become diverging. Which can happen but two ways only: for as they go on increasing still the more, as they accede to the Point of Inflexion or Retrogression, they must become at last parallel, that is, the Radius of Evolution will be infinite: and where they constantly continue decreasing, they must at last coincide, that is, the Radius of Evolution will be nothing. All this is correspondent to what has been demonstrated in the foregoing Section.

SCHOLIUM.

82. **B**ECAUSE hitherto the Radius of Evolution has been consider'd as infinitely great in the Point of Inflexion; I shall here shew, that in numberless Species of Curves the Radius of Evolution in the Point of Inflexion is equal to nothing: and that there is but one Species where the said Radius is infinite.

Let BAC be a Curve, whose Radius of Evolution in the Point of Inflexion A is infinite. FIG. 71.

Now if the Parts BA, AC be consider'd as evolute Curves, the Point A being supposed their Beginning, and the Curve DAE as an involute Curve formed from them: it is plain that this latter Curve will have a Point of Inflexion in A , but the Radius of Evolution in that Point will be equal to nothing. And if a third involute Curve be formed from the second DAE as an Evolute, and a fourth Involute

lute from the third as an Evolute, and so on; it is manifest that the Radius of Evolution in the Point of Inflexion A in every of those Curves, will always be equal to nothing. Whence, &c.

PROP. II.

FIG. 72. 83. **L**ET AMD be an involute Curve, whose Axis AB is at right Angles to the Tangent in A : to find the Point B wherein the said Axis touches the Evolute BCG .

If the Point M be supposed to become infinitely near the Vertex A , it is plain that the Perpendicular MQ will intersect the Axis in the Point B sought. Whence if you seek the Value in general of $PQ \left(\frac{y\dot{y}}{\dot{x}} \right)$ in x or y , and afterwards you make x or $y = 0$, we may determine the Point P , which is to coincide with A , and the Point Q which coincides with the Point B sought; that is, PQ will then become equal to AB sought. This will be more plain by the following Examples.

EXAMPLE I.

FIG. 72. 84. **L**ET the Involute AMD be a Parabola, whose Parameter is a right Line, suppose a . The Equation of the Curve is $ax = yy$, the Fluxion whereof is $\dot{y} = \frac{a\dot{x}}{2y} = \frac{a\dot{x}}{2\sqrt{ax}}$; and throwing this last Equation again into Fluxions, by making x invariable, there arises $\dot{y} = \frac{-a\dot{x}^2}{4x\sqrt{ax}}$. Now substituting these Values for $\dot{y}$ and

$\dot{y}$ and $\ddot{y}$ in the general Form $\frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}}$, and then

will * $ME = \frac{a + 4x\sqrt{ax}}{a} = \sqrt{ax} + \frac{4x\sqrt{ax}}{a}$. * Art. 77.

From whence arises the following Construction.

From the Point T , wherein the Tangent TM intersects the Axis, draw TE parallel to MC : I say, this shall meet MP continued out in the Point E sought. For because of the right Angles MPT , MTE , $MP(\sqrt{ax})$:

$$PT(2x) :: PT(2x) : PE = \frac{4xx}{\sqrt{ax}} = \frac{4x\sqrt{ax}}{a};$$

and consequently $MP + PE = \sqrt{ax} + \frac{4x\sqrt{ax}}{a}$.

Again, because of the right-angled Triangles MPQ , MEC , therefore $PM(\sqrt{ax}) : PQ$

$$(\frac{1}{2}a) :: ME\left(\sqrt{ax} + \frac{4x\sqrt{ax}}{a}\right) : EC \text{ or } PK =$$

$\frac{1}{2}a + 2x$. And therefore $QK = 2x$. From whence we get this new Construction.

Take QK the Double of AP , or (which is the same) take $PK = TQ$, and draw KC parallel to PM . This will meet the Perpendicular MC in the Point C , which will be in the Evolute BCG .

Otherwise. $yy = ax$, and $2y\dot{y} = a\dot{x}$ (and taking $\dot{x}$ as invariable) $2\dot{y}^2 + 2y\ddot{y} = 0$; whence there arises

$$-\ddot{y} = \frac{\dot{y}^2}{y}. \text{ And putting this Value in the Form}$$

$$\frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}}, \text{ there will arise } * ME = \frac{y\dot{y}^2 + y\dot{x}^2}{\dot{y}^2}; * \text{ Art. 77.}$$

$$\text{and therefore } EC \text{ or } PK = \frac{y\dot{y}^2 + y\dot{x}^2}{\dot{y}\dot{x}} = \frac{y\dot{y}}{\dot{x}} + \frac{y\dot{x}}{\dot{y}}$$

$$= PQ + PT \text{ or } TQ. \text{ And so we get the same}$$

same Constructions as before. For $MP:PT :: \dot{y} : \dot{x} :: PT \left(\frac{y \dot{x}}{\dot{y}} \right) :: PE = \frac{y \dot{x}^2}{\dot{y}^2} = \frac{4x\sqrt{ax}}{a}$.

Now to find the Point B , wherein the Axis AB touches the Evolute BCG : we have $PQ \left(\frac{y \dot{y}}{x} \right) = \frac{1}{2}a$. And since this is invariable, where-ever the Point M is assumed; therefore when M coincides with the Vertex A , we still shall have PQ , which in this Case becomes $AB = \frac{1}{2}a$.

To find the Nature of the Involute BCG according to *Descartes's* way. Call the Absciss BK, u ; and the Ordinate KC or PE, t ; then will $CK(t) = \frac{4x\sqrt{ax}}{a}$, and $AP + PK - AB(u) = 3x$. Now putting $\frac{1}{3}u$ for its Equal x in the Equation $t = \frac{4x\sqrt{ax}}{a}$, and there

will come out $27att = 16u^3$ expressing the Relation of BK to KC . Whence if the Involute be a common Parabola, the Evolute BCG is a second cubical Parabola, whose Parameter is equal to $\frac{27}{16}$ of the Parameter of the Involute.

FIG. 73. It is evident, that if the Involute be the whole Parabola MAM , the Evolute CBC will consist of two Parts CB, BC , having contrary Concavities; so that B will be a Point of Retrogression.

DEFINITION.

FIG. 72. *BY Geometrical Curves, as AMD, BCG, I understand those, whereof the Relation of the Abscisses AP, BK to the correspondent Ordinates*

nates PM , KC , being all right Lines, can be expressed by a finite Equation free from Fluxions. And whatever is effected by means of those Lines, is said to be Geometrical.

C O R O L L.

85. **W**HEN the given Involute Curve AMD is a Geometrical one, it is plain that we can (as above) always find an Equation expressing the Nature of the Evolute BCG ; and so the Evolute will be likewise a Geometrical Curve. I say moreover that it is to be rectified, viz. we can find geometrically straight Lines equal to the Length of any Part BC of it. For * by means of the Involute AMD , * Art. 75. which is a Geometrical Curve, we can determine the Point M in the Tangent CM to the Part BC , such that the straight Line CM differs from the Part BC only by a given right Line AB .

E X A M P L E II.

86. **L**ET the Involute MDM be an Hyperbola within its Asymptotes. The Nature of this is $aa = xy$. FIG. 74.

Now $\frac{aa}{y} = x$, $\frac{aa\dot{y}}{yy} = \dot{x}$, and supposing $\dot{x}$ invariable * $\frac{-aay\ddot{y} + 2aay\dot{y}^2}{y^4} = 0$; whence * Art. 77.

we get $\ddot{y} = \frac{2\dot{y}^2}{y}$, and putting this Value in $\frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}}$,

there comes out * $ME = \frac{y\dot{x}^2 + y\dot{y}^2}{-2\dot{y}^2}$: so that EC * Art. 77.

or $PK = -\frac{y\dot{y}}{2\dot{x}} - \frac{y\dot{x}}{2y}$. Hence the following

Constructions are derived.

Thro'

Thro' the Point T wherein the Tangent MT intersects the Asymptote AB , draw TS parallel to MC meeting MP continued out in S ; and assume ME equal to $\frac{1}{2} MS$ on the other side the Asymptote (which is here taken as the Axis) because the Value of it is negative; or else assume PK equal to $\frac{1}{2} TQ$, on the same side as the Point T is: then, I say, if EC be drawn parallel, or KC perpendicular to the Axis, they will intersect MC in the Point C sought: for it is plain that $MS = \frac{y\dot{x}^2 + \dot{y}y^2}{\dot{y}^2}$, and $TQ = \frac{y\dot{y}}{\dot{x}} + \frac{y\dot{x}}{\dot{y}}$.

From an Inspection of the Figure of the Hyperbola MDM , it will not be difficult to perceive that the Evolute CLC must have a Point L of Retrogression, as the Evolute in the last Example has. To determine which, we must observe that the Radius of Evolution DL is less than any other Radius MC ; so that the Fluxion of the Expression thereof,

* Art. 78. viz. $\frac{\dot{x}^2 + \dot{y}^2 \sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\dot{y}}$ or $\overline{\dot{x}^2 + \dot{y}^2}^{\frac{3}{2}}$ will be * nothing or infinite. And so taking $\dot{x}$ as invariable, the second Fluxion will be

$$-\frac{3\dot{x}\dot{y}\ddot{y}^2\dot{x}^2 + \dot{y}^2 + \dot{x}\ddot{y}\dot{x}^2 + \dot{y}^2}{\dot{x}^2\dot{y}^2} = 0, \text{ or infi-}$$

nite; whence dividing by $\overline{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}}$, and afterwards multiplying by $\dot{x}\dot{y}^2$, there arises $\dot{x}^2\dot{y} + \dot{y}^2\ddot{y} - 3\dot{y}\ddot{y}^2 = 0$, or infinite; by which we may find such an Expression as AH for x , that by drawing the Ordinate HD and the Radius of Evolution DL , the Point L will be the Point of Retrogression sought.

of FLUXIONS.

III

In this Example $y = \frac{aa}{x}$, $\dot{y} = \frac{-aa\dot{x}}{xx}$, $\ddot{y} = \frac{2aa\dot{x}^2}{x^3}$, $\ddot{\dot{y}} = \frac{-6aa\dot{x}^3}{x^4}$. Whence putting

the latter Members of every of these Equations for the former ones in the Equation aforegoing, and there comes out $AH(x) = a$. Therefore the Point D is the Vertex of the Hyperbola, and the Lines AD , DL coincide with AL . Which is the Axis of the Curve.

EXAMPLE III.

87. **L**ET $y^m = x$ generally express the Nature of all Parabola's, the Exponent m representing a positive whole Number or Fraction, and all Hyperbola's, when the same Exponent is a negative whole Number or Fraction. FIG. 72, 74.

Now $my^{m-1}\dot{y} = \dot{x}$, and the Fluxion of this again, taking $\dot{x}$ as invariable, will be $mm - my^{m-2}\dot{y}^2 + my^{m-1}\ddot{y} = 0$; and dividing by my^{m-1} , there comes out $-\ddot{y} = \frac{m-1\dot{y}^2}{y}$; whence substituting this latter Expression in $\frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}}$, we* get $ME = \frac{y\dot{x}^2 + \dot{y}y^2}{m-1\dot{y}^2}$; * Art. 77.

and therefore EC or $PK = \frac{y\dot{y}}{m-1\dot{x}} + \frac{y\dot{x}}{m-1\dot{y}}$.

From whence the following Constructions are gained.

From T the Intersection of the Axis AP , with the Tangent MT draw the Line TS parallel to MC meeting MP continued out in the Point S : Assume $ME = \frac{1}{m-1} MS$, or
else

else take $PK = \frac{1}{m-1} TQ$; then if a paral-

lel to the Axis be drawn thro' E , or a Perpendicular thro' K ; these shall intersect MC in the Point C sought.

FIG. 74. If m be negative, as in the Hyperbola's, the Value of ME will be negative; and so the Curves will be convex towards their Axis,

FIG. 75. which will be an Asymptote then. But in the Parabola's where m is positive, there may happen two Cases: for when m is less than 1, they will be convex next to the Axis, which will be a Tangent to the Vertex; and when

FIG. 72. m exceeds 1, then they will be concave next to the Axis, which will be perpendicular in the Vertex.

Now in this latter Case, to find the Point B where the Axis AB touches the Evolute.

$PQ \left(\frac{yy'}{x} \right) \text{ is } = \frac{y^2 - m}{m}$; from whence arises

three Cases. For when $m = 2$, as in the common Parabola, then the Exponent of y being nothing, that unknown Quantity vanishes; and consequently $AB = \frac{1}{2}$, viz. to one half the Parameter. When m is less than 2, then the Exponent of y being positive, it shall be found in the Numerator, and the Fraction will be-

* Art. 83. come nothing by making * it equal to 0; that is, the Point B in this Case will coincide with A , as in the second Cubick Parabola $axx = y^3$.

FIG. 76. Or lastly, when m is greater than 2, and then the Exponent of y being negative, it will be in the Denominator; so that when it becomes nothing, the Fraction will be infinite: viz. the Point B is infinitely distant from A , or (which is the same) the Axis AB is an Asymptote of the Evolute, as in the first Cubick

Para-

Parabola $axx=y^2$. In this latter Case, we may observe that the Involute CLO , when Fig. 77. the Evolute ADM is a Semi-parabola, will have a Point of Retrogression; so that tho' the Part LO of the Evolute be infinite, yet the Part DA of the Involute formed by it will be determinate or finite; and the other Part LC of the Evolute being infinite, will notwithstanding describe the infinite Part DM of the Involute.

Now the Point L may be determined after the same way as in the Hyperbola. For Ex-

ample: Let $axx=y^2$, or $y=x^{\frac{1}{2}}$; then will $\dot{y}=\frac{1}{2}x^{-\frac{1}{2}}$, $\ddot{y}=-\frac{1}{4}x^{-\frac{3}{2}}$, $\ddot{\dot{y}}=\frac{3}{8}x^{-\frac{5}{2}}$, which Values being substituted in the Equation $\dot{x}^2\ddot{y}+\dot{y}^2\ddot{x}-3\dot{y}\ddot{y}^2=0$, and there comes

$$* \text{ out } AH(x) = \sqrt[3]{\frac{1}{91125}}$$

* Art. 86

SCHOLIUM.

88. **W**HEN m is greater than 1; and so the Parabola's concave next to the Axis, there are several Cases. For if the Numerator of the Fraction denoted by m be even, and the Denominator odd, all the Parabola's will fall on each Side of their Axis just as the common Parabola does. But if the Numerator and Denominator be each odd, they have an inverted Position on each Side their Axis, so that the Vertex A is a Point of Inflexion; as the first Cubical Parabola $x=y^{\frac{1}{3}}$, or $axx=y^2$. Fig. 77. Lastly, if the Numerator be odd, and the Denominator even, these Curves have an inverted Position on the same Side their Axis; Fig. 76.

L so

so that the Vertex A is a Point of Retrogression; as the second Cubical Parabola $x = y^{\frac{2}{3}}$ or $axx = y^2$. The Reason of all this is, because an even Power cannot have a negative Value. Whence it evidently follows,

FIG. 77. 1^o, That in the Point of Inflexion A , the Radius of Evolution may be infinitely great, as in $axx = y^2$; or infinitely small, as in $axx^2 = y^3$.

FIG. 76. 2^o, That in the Point of Retrogression A , the Radius of Evolution may be either infinite, as in $a^2xx = y^3$; or nothing, as in $axx = y^2$.

FIG. 73. 3^o, That because the Radius of Evolution is infinite or 0, it does not from hence follow that the Curves have then a Point of Inflexion or Retrogression: for in $a^2x = y^4$ it is infinite, and in $ax^3 = y^4$ it is 0; and yet these Parabola's have the same Position with regard to their Axis, as the common Parabola.

EXAMPLE IV.

FIG. 78, 79. **L**ET the Involute AMD be an Hyperbola or Ellipsis, whose Axis is $AH(a)$, and Parameter $AF(b)$.

Then from the Nature of these Curves $y = \sqrt{\frac{abx + bxx}{a}}$, $\dot{y} = \frac{abx + 2bxx}{2\sqrt{aabx + abxx^2}}$, and $\ddot{y} = \frac{-a^3bbx^2}{4aabx + 4abxx\sqrt{aabx + abxx}}$. Now if these Values be introduced into $\frac{\dot{x}^2 + \dot{y}^2 \sqrt{x^2 + y^2}}{\dot{x}\ddot{y}}$, the ge-

* Art. 78. neral Expression for MC ; then will $MC = \frac{aab + 4abbx + 4bbxx + 4aabx + 4abxx\sqrt{aab + 4abbx + 4bbxx + 4aabx + 4abx^2}}{2a^3bb}$
 $= \frac{4M^2}{bb}$, because on each Side $M^2 \left(\frac{y\sqrt{x^2 + y^2}}{\dot{x}} \right) =$

$\sqrt{aa}$

$$\sqrt{aabb + 4abbx + 4bbxx + 4aabbx + 4abxx} = 2a$$

Hence we get the following Construction, which will serve for the Parabola also.

Assume MC equal to four times a fourth continual Proportional between the Parameter AF , and the Perpendicular MQ bounded by the Axis: then will the Point C fall in the Evolute.

If you make $x = 0$, then will $* AB = \frac{1}{2}b$. * Art. 83. And if in the Ellipsis you make $x = \frac{1}{2}a$; then FIG. 79. will DG be $= \frac{a\sqrt{ab}}{2b}$, viz. equal to half the

Parameter of the conjugate Axis. And so it appears that the Evolute BCG , when the Involute is an Ellipsis, terminates in the Point G of the conjugate Axis DO ; wherein is formed a Point of Retrogression; but in the Parabola and Hyperbola it runs out *ad infinitum*.

If $a = b$ in the Ellipsis, there comes out $MC = \frac{1}{2}a$; from whence it follows, that the Radii of Evolution are all equal to one another; and so the Evolute will become here a Point; that is, the Involute will be a Circle, and the Evolute the Centre of it. Which is an establish'd Truth from other Principles.

EXAMPLE V.

90. **L**ET AMD be a Logarithmick Curve FIG. 80. of such a Nature, that if from any Point M in the same be drawn the Perpendicular MP to the Asymptote KP and the Tangent MT ; the Subtangent be $PT = a$ a given right Line.

Now $PT\left(\frac{y\dot{x}}{\dot{y}}\right) = a$; whence we get $\dot{y} = \frac{y\dot{x}}{a}$, and the Fluxion of this again, supposing $\dot{x}$ invariable, will be $\ddot{y} = \frac{\dot{y}\dot{x}}{a} = \frac{y\dot{x}^2}{aa}$; and putting

• *Art. 77.* these Values in $\frac{\dot{x}^2 + \dot{y}^2}{-\dot{y}}$; and there will arise *

$$ME = \frac{-aa - yy}{y}; \text{ and therefore } EC \text{ or } PK = \frac{-aa - yy}{y}.$$

From whence arises this Construction.

Assume $PK = TQ$ on the same Side T , because the Value of it is negative; and draw KC parallel to PM : I say the same will intersect the Perpendicular MC in the Point C sought. For $TQ = \frac{aa + yy}{a}$.

If you have a mind that the Point M be in the greatest Part possible of the Curvature, the general Expression $\dot{x}^2\ddot{y} + \dot{y}^2\ddot{x} - 3\dot{y}\ddot{x}\dot{y} = 0$, (of *Art. 86.*) must be used: then if $\frac{y\dot{x}}{a}, \frac{y\dot{x}^2}{aa}, \frac{y\dot{x}^3}{a^3}$ be put for $\dot{y}, \ddot{y}, \ddot{\dot{y}}$, $PM(y)$ will be $= a\sqrt{\frac{1}{3}}$.

Now when $\dot{x}$ is invariable, it is plain that the Ordinates y are to each other as their Fluxions $\dot{y}$ or $\frac{y\dot{x}}{a}$; from whence it follows, that they will be in a geometrical Progression also. For if the Asymptote, or Axis PK , be supposed to be divided into an infinite Number of small equal Parts Pp or MR , pf or mS , fg or nH , &c. intercepted by the Ordinates PM , pm , fn , go , &c. then will $PM:pm::Rm:Sn::Rm::PM + Rm$, or $pm:pm + Sn$ or Fn .

In like manner we prove that $pm:fn::fn:go$, and so on. Therefore the Ordinates PM , pm , fn , go , &c. are in a geometrical Progression.

EXAMPLE VI.

91. **L**ET AMD be a Logarithmetical Spi- Fig. 81.
ral, of such a Nature, that if you draw a right Line MA from any Point M in it, to the Point A being the Centre, and the Tangent MT ; the Angle AMT is of a given Quantity, viz. always the same.

Because the Angle AMT or AmM is invariable, the Ratio of $mR(\dot{y})$ to $RM(\dot{x})$ will be also invariable; therefore the Fluxion of $\frac{\dot{y}}{\dot{x}}$ must be nothing; so that (supposing $\dot{x}$ invariable) $\ddot{y} = 0$. Therefore if the Term $y\ddot{y}$ be struck out of $\frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 + \dot{y}^2 - y\ddot{y}}$ the general Expression * for ME , when the Ordinates issue from * Art. 77. the same Point, ME will be $=y$, viz. $ME = AM$. From whence comes the following Construction.

Draw AC perpendicular to AM , meeting MC the Perpendicular to the Curve in C , which will be a Point in the Evolute ACB .

The Angles AMT , ACM are equal, because by adding AMC to each of them, the wholes will be right Angles. Therefore the Evolute ACG will be a Logarithmick Spiral differing only from the Involute AMD in Position.

If the Point C in the Evolute ACG be given, and it be requir'd to find the Length of CM the Radius of the Evolute in that Point, which is * equal to the Part AC of the Spiral * Art. 75.
I 3 making

making an infinite Number of Revolutions before its Accession to A ; it is manifest, that AM need only be drawn perpendicular to AC . So that if AT be drawn perpendicular to AM , the Tangent MT will be equal likewise to the Part AM of the Logarithmick Spiral AMD given.

If you imagine an infinite Number of Ordinates $AM, Am, An, Ao, \&c.$ making infinitely small equal Angles; it is manifest, that the Triangles $MAM, mAn, nAo, \&c.$ will be similar, because the Angles at A are equal, as likewise are the Angles at $m, n, o, \&c.$ from the Nature of the Spiral. Therefore $AM:Am::Am:An$; and $Am:An::An:Ao, \&c.$ Whence the Ordinates $AM, Am, An, Ao, \&c.$ are in a geometrical Progression, if they make equal Angles with each other.

EXAMPLE VII.

FIG. 82. 92. **L**ET AMD be one Spiral (of an infinite Number) formed in the Sector BAD , of such a Nature, that any Radius AMP being drawn, and calling the whole Arch BPD, b ; the Part BP, z ; the Radius AB or AP, a ; and the Part AM, y ; there is always the following Proportion, $b:z::a^m:y^m$.

The Equation of the Spiral AMD is $y^m = \frac{a^m z}{b}$, which thrown into Fluxions is $my^{m-1} \dot{y} = \frac{a^m \dot{z}}{b}$.

Now because of the similar Sectors $AMR, APp, AM(y):AP(a)::MR(\dot{x}):Pp(\dot{z}) = \frac{a\dot{x}}{y}$. Now putting this Value for $\dot{z}$

in the Equation afore found, and $my^m \dot{y} = \frac{a^m \dot{x}}{y}$

$\frac{a^{m-1}\dot{x}}{b}$; which again being thrown into Fluxions making $\dot{x}$ invariable, and $mmy^{m-1}\dot{y}^2 + my^m\ddot{y} = 0$; whence dividing by my^{m-1} , we get $-y\ddot{y} = m\dot{y}^2$; and therefore * *ME* * Art. 77.

$$\left(\frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 + \dot{y}^2 - y\ddot{y}} \right) = \frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 + m + 1\dot{y}^2}. \text{ So that the}$$

following Construction comes out thus.

Thro' *A* draw *TAQ* perpendicular to *AM* meeting the Tangent *MT* in *T*, and the Perpendicular *MQ* in *Q*; make *TA + m + 1AQ*: *TQ*:*MA*:*ME*. Then if *EC* be drawn parallel to *TQ*, it will meet *MQ* in a Point, as *C*, which will be in the Evolute.

For because *MRG*, *TAQ* are parallel,

$$MR(\dot{x}) + \overline{m+1}RG\left(\frac{\dot{y}^2}{\dot{x}^2}\right):MG\left(\dot{x} + \frac{\dot{y}^2}{\dot{x}}\right)$$

$$::TA + \overline{m+1}AQ:TQ::AM(y):ME =$$

$$\frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 + \overline{m+1}\dot{y}^2}.$$

EXAMPLE VIII.

93. **L**ET *AMD* be Semi-cycloid, whose Base *BD* is equal to $\frac{2}{3}$ the Circumference *BEA* of the generating Circle. FIG. 83.

Call *AP*, *x*; *PM*, *y*; the Arch *AE*, *u*; and the Diameter *AB*, *2a*. Then *PE* = $\sqrt{2ax - xx}$, from the nature of the Circle, and *y* = *u* + $\sqrt{2ax - xx}$, from the nature of the Cycloid; which last Equation thrown into Fluxions will

$$\text{be } \dot{y} = \dot{u} + \frac{a\dot{x} - x\dot{x}}{\sqrt{2ax - xx}} = \frac{2a\dot{x} - x\dot{x}}{\sqrt{2ax - xx}}, \text{ or}$$

$$\dot{x} \frac{\sqrt{2a-x}}{x}, \text{ putting } \frac{a\dot{x}}{\sqrt{2ax - xx}} \text{ for } \dot{u}; \text{ and supposing}$$

x invariable, $y = \frac{-ax^2}{x\sqrt{2ax-xx}}$; and substitu-

ting these Expressions in $\frac{x^2+y^2\sqrt{x^2+y^2}}{-xy}$, and

Art. 78. there comes * out $MC = 2\sqrt{4aa-2ax}$, that is, $2BE$ or $2MG$.

If you make $x=0$; then will $AN=4a$ be the Radius of Evolution in the Vertex A . But if $x=2a$, the Radius of Evolution in D will become nothing. From whence we are assured, that D is the Beginning of the Evolute, and the Point N the End, so that $BN=BA$.

Now to find the Nature of this Involute, you need only compleat the Rectangle BS , describe the Semicircle DIS on DS , as a Diameter, and draw DI parallel to MC or BE . This being done, it is plain that the Angle BDI is equal to the Angle EBD ; and consequently the Arches DI, BE are equal to one another; whence the Chords DI, BE or GC are likewise equal. And therefore if IC be drawn, it will be equal and parallel to DG , which by the Generation of the Cycloid is equal to the Arch BE or DI ; and consequently the Evolute DCN is a Semi-cycloid, having the right Line NS for its Base, being equal to $\frac{1}{2}$ the Circumference DIS of the generating Circle: that is, it will be the same Semi-cycloid $AMDB$ having an inverted Situation.

COROL.

COROL.

94. **I**T is * evident that the Part DC of the * Art. 75.
Cycloid is the Double of its Tangent
 CG , or correspondent Chord DI ; and the
Semi-cycloid DCN , is the Double of the
Diameter BN or DS of the generating Cir-
cle.

ANOTHER SOLUTION.

95. **T**HE Length of the Radius MC may be
found without any Calculus, thus:
Conceive the Perpendicular mC to be infinitely
near the former one, another Parallel me , ano-
ther Chord Be , and from the Centres C, B ,
describe the small Arches GH, EF ; then
will the right-angled Triangles GHg, EFe ,
be equal and similar; for $Gg = Ee$, since BG
or ME is equal to the Arch AE , and in like
manner, Bg or me is equal to the Arch Ae ;
moreover Hg or $mg - MG = Fe$ or $Be - BE$;
whence GH will be equal to EF . Now since
the Perpendiculars MC, mC , are parallel to
the Chords EB, eB , the Angle MCm shall
be equal to the Angle EBe . Therefore since
the Arches GH, EF , that are the Measures of
those Angles, are equal, it follows that the
Radii CG, BE shall be equal likewise; and so
 MC must be assumed the Double of MG or
 BE .

LEMMA.

96. **I**F there be any Number of Quantities $a, b,$
 $c, d, e, \&c.$ either finite or infinite, the
Sum of their Differences, viz. $a - b + b - c +$
 $c - d + d - e, \&c.$ is equal to the greatest Quan-
tity

tity a Minus the least e , or else equal to the greatest, when the least is 0.

COROL. I.

97. **B**ECAUSE the Sectors CMm , CGH are similar, Mm is $= 2GH = 2EF$; and since this is always so, be the Point M where it will; therefore the Sum of all the little Arches Mm , that is, the Part AM of the Semi-cycloid AMD , is the Double of the Sum of all the little Arches EF . But the little Arch EF , being a Part of the Chord AE perpendicular to BE , is the Difference of the Chords AE , Ae , since the small right Line eF perpendicular to Ae , may be consider'd as a small Arch described about the Centre A . And therefore the Sum of all the little Arches EF in the Arch AZE , will be the Sum of the Differences of all the Chords AE , Ae , &c. in that Arch; that is (by the Lemma above) equal to the Chord AE . Therefore it is evident, that the Part AM of the Semi-Cycloid AMD , is the Double of the correspondent Chord AE .

COROL. II.

* Art. 2. 98. **T**HE SPACE $MGgm^*$, or the Trapezium $MGHm = \frac{1}{2}Mm + \frac{1}{2}GH \times MG = \frac{1}{2}EF \times BE$; that is, it is the Triple of the Triangle EBF or EBe ; whence the Space $MGBA$, or Sum of all the said Trapezia, is the Triple of the circular Space $BEZA$, being the Sum of all the said Triangles.

EXAM-

EXAMPLE III.

99. **C**ALL BP, z ; the Arch AZE or EM or BG, u ; and the Radius KA, a ; then will the Parallelogram $MGBE$ be $=uz$. Now the cycloidal Space $MGBA = 3BEZA = 3EKB + \frac{1}{2}au$; and therefore the Space $AMEB$ contained under the Part AM of the Cycloid, the Parallel ME , the Chord BE , and the Diameter AB , is $= 3EKB + \frac{1}{2}au - uz$. Whence if you assume $BP (z) = \frac{1}{3}a$, the Space $AMEB$ will be the Triple of the correspondent Triangle EKB ; and so we have the Quadrature thereof independent of the Quadrature of the Circle; (which Mr. *Hugens* first observed) as likewise the Quadrature of the following Space.

If the Segment $BEZA$ be taken away from the Space $AMEB$, there will remain the Space $AZEM = 2EKB + au - uz$; therefore when the Point P coincides with K , the Space $AZEM$ will be then equal to the Square of the Radius.

It is plain, that among all the Spaces $AMEB$, $AZEM$, there are but these two only that can be squared independent of the Quadrature of the Circle.

EXAMPLE IX.

100. **L**ET AMD be a Semi-cycloid descri- FIG. 84.
bed by the Rotation of the Semi-circle AEB about or along another Circle BGD at rest. It is requir'd to find that Point in the Perpendicular MG of a given Position that touches the Evolute.

Because

Because in the Use of general Expressions or Formules, we must first take right Lines perpendicular to the Axis AO , as Ordinates to the Curve AMD , and then find an Equation expressing the Relation of the Ordinates to the Abscisses, or of their Fluxions, which often is a very operose Performance; therefore in Occurrences of this Kind, it is much better to endeavour at the Solution from the Generation itself.

When the Semi-circle AEB is come to the Situation MGB , wherein it touches the Base DB in G , and the describent Point A , falls on the Point M of the Semi-cycloid AMD , it is plain,

1^o, That the Arch GM is equal to the Arch GD ; as also the Arch GB of the moveable Circle, to the Arch GB of the immoveable one.

• Art. 43. 2^o, That MG is * perpendicular to the Curve : For if you consider the Semi-circumference MGB or AEB , and the Base BGD , as the Assemblage of an infinite Number of little equal right Lines, each equal to its Correspondent, it is manifest, that the Semi-cycloid AMD will be the Assemblage of an infinite Number of little Arches, whose Centres are successively the Contact Points G , each being described thro' the same Point M or A .

3^o, That if from O the Centre of the immoveable Circle, the Arch ME be described, then the Arches MG , EB , of the moveable Circle, will be equal to one another, as well as their Chords MG , EB , and the Angles OGM , OBE . For the Lines OK , OK , joining the Centres of the Circles are equal, because they pass thro' the Points of Contact B , G . Therefore drawing the Ra-

dii OM , OE , and KE , the Triangles OKM , OKE , will be equal and similar. Whence because the Angle OKM is equal to the Angle OKE , the Arches MG , BE , of the equal Semi-circles MGB , BEA , being the Measures of these Angles, shall be equal; as likewise their Chords MG , EB ; and so the Angles OGM , OBE , are equal likewise.

This being laid down, let mC be another FIG. 85.
Perpendicular infinitely near the first, me another concentrick Arch, and Be another Chord; and from the Centres C and B , describe the small Arches GH , EF . Now the right-angled Triangles GHg , EFe , are equal and similar; for Gg or $Dg - DG = Ee$, or the Arch Be — the Arch BE : Moreover, Hg or $mg - MG = Fe$ or $= Be - BE$. Whence the little Arch GH will be equal to the little Arch EF ; and so the Angle GCH , is to the Angle EBF , as BE to CG . And therefore the whole Difficulty is brought to this, viz. to find the Relation between those Angles; which may be effected thus:

Draw the Radii OG , Og , KE , Ke , and call OG or OB , b ; KE or KB , or KA , a ; then it is manifest, that the Angle $EBe = OBe - OBE = Og m - OGM =$ (drawing GL , GV , parallel to Cm , Og) $LGM - OG V = GCH - GOg$. Therefore the Angle GCH shall be $= GOg + EBF$. But because the Arches Gg , Ee , are equal, $GOg : EKe$ or $2EBF :: KE(a) : OG(b)$; and consequently the Angle $GOg = \frac{2a}{b} EBF$, and $GCH = \frac{2a+b}{b} EBF$. Whence $GCH : EBF$ or BE

$:: CG$

$\therefore CG : \frac{2a+b}{b} : 1$. And so the unknown Quantity $CG = \frac{b}{2a+b} BE$ or MG .

FIG. 86. Whence if you make $OA(2a+b) : OB(b) :: MG : GC$, the Point C will be in the Evolute.

It is evident, 1^o, That the Evolute begins at the Point D , in which it touches the Base BGD , because the Arch GM becomes infinitely small in that Point. 2^o, And that it ends in the Point N ; so that $OA : OB :: AB : BN :: OA - AB$ or $OB : OB - BN$ or ON ; that is, OA, OB, ON , are continual Proportionals. 3^o, If the Circle NSQ be described from the Centre O , then, I say, the Evolute DCN is generated by the Rotation of the moveable Circle GCS , having GS or BN , as a Diameter, along the immoveable one NSQ : That is, the said Evolute is a Semi-cycloid, (because the Diameters AB, BN , of the moveable Circles, are to one another, as the Radii OB, ON , of the immoveable one) having an inverted Situation with respect to that of the other, the Vertex being in D .

To prove this, let us suppose the Diameters of the moveable Circles to fall in the right Line OT drawn at pleasure from the Centre O ; this shall pass thro' the Points of Contact S, G ; and making AB or $TG : BN$ or $GS :: MG : GC$, the Point C will be in the Evolute, as likewise in the Circumference of the Circle GCS ; for the Angle GMT being a right Angle, the Angle GCS shall be so likewise. But because of the equal Angles MGT, CGS , the Arch TM or GB is to the Arch CS , as the Diameter GT to the Diameter $GS :: OG : OS :: GB$

$\therefore GB:NS$; and therefore the Arches CS, SN , are equal. Whence, &c.

COROL. I.

101. **I**T is * plain that the Part DC of the *Art. 75.*
Cycloid is equal to the right Line CM ,
and therefore DC is to the Tangent $CG:AB$
 $+BN:BN::OB+ON:ON$; that is, as the
Sum of the Diameters of the generating Cir-
cles, or of the moveable and immoveable ones,
is to the Radius of the immoveable Circle.
The Truth of this may be shewn otherwise,
thus:

Because of the Similarity of the Triangles *Fig. 85.*
 $CMm, CGH, Mm:GH$ or $EF::MCGC::OA$
 $+OB, (2a+2b):OB(b)$. Whence (as in *Art.*
97.) the Part AM of the Cycloid is to the cor-
respondent Chord AE , as the Sum of the Di-
ameters of the generating Circle, and the Base
is to the Semi-diameter of the Base.

COROL. II.

102. **T**HE Trapezium $MGHm = \frac{1}{2}GH + \frac{1}{2}Mm$ *Fig. 85.*

$\times MG$. Now $CG \left(\frac{b}{2a+b} MG \right):$
 $CM \left(\frac{2a+2b}{2a+b} MG \right)::GH:Mm =$
 $\frac{2a+2b}{b} GH$. Therefore because $GH=EF$,

and $MG=EB$, $MGHm$ shall be $= \frac{2a+3b}{2b}$

$EF \times EB$. That is, the Trapezium $MGHm$
will be always to the correspondent Triangle
 $EBF::2a+3b:b$.

Whence

Whence the Space $MGBA$ contained under MG , AB , Perpendiculars to the Cycloid, the Arch BG , and the Part MA of the Curve, is to the correspondent Segment $BEZA$ of the Circle, as $2a+3b$ is to b .

COROL. III.

FIG. 87. 103. IT is evident that the Quadrature of any

Part of the Cycloid depends on the Quadrature of the Circle; but if OQ be taken as a mean Proportional between OK , OA , and the Arch QEM being described with that as a Semi-diameter. I say, the Space $ABEM$, contained under the Diameter AB , the Chord BE , the Arch EM , and the Part AM of the Cycloid, is to the Triangle EKB :: $2a+3b:b$.

For call the Arch AE or GB , u ; and the Radius OQ , z ; then will $OB(b):OQ(z)::GB(u):RQ$ or $ME = \frac{uz}{b}$. And therefore

the Space $RGBQ$ or $MBGE$, that is,

$$\frac{1}{2}GB + \frac{1}{2}RQ \times BQ = \frac{zzu - bbu}{2b}.$$

Now the Space

*Art. 102. * $MGBA = \frac{2a+3b}{b} \times BEZA = \frac{2a+3b}{b} \times EKB + \frac{2a+3b}{b} \times KEZA \left(\frac{au}{2} \right)$. Now if the Space

aforesaid be taken from this, there shall remain

$$ABEM = \frac{2aa + 3abu + bbu - zz u}{2b} + \frac{2a+3b}{b}$$

$\times EKB = \frac{2a+3b}{b} EKB$; because by Constru-

ction $zz = 2aa + 3ab + bb$. Whence the said Space is the only one, among others like it, whose Quadrature is independent of that of the Circle.

Moreover,

Moreover, we may have another mixed-lined Space, whose Quadrature is independent of that of the Circle. For if from the Space $ABEM$, the Segment $BEZA$ ($\frac{1}{2}au + EKB$) be taken away, there will remain $AZEM = \frac{2aan + 2abn + bbn - zzu}{2b} + \frac{2a+2b}{b} \times EKB = \frac{2a+2b}{b} EKB$, by making $zz = 2aa + 2ab + bb$: That is, if the Semi-circumference be bisected in the Point E , the Space $AZEM$ shall be to the Double of the Triangle EKB , viz. to the Square of the Semidiameter $\therefore OK(a+b) : OB(b)$.

COROL. IV.

104. **W**HEN the moveable Circle AEB revolves within the immoveable one BGD , the Diameter AB thereof, which before was positive, is here negative; and therefore the Signs of the Terms affected with it when its Dimensions are odd, must be changed. Whence,

1^o, If you draw MG at pleasure perpendicular to the Cycloid, and make $OA(b-2a) : OB(b) :: MG : GC$, the Point C will be * in the Evolute DCN described by the Rotation of the Circle (having BN as a Diameter) within the Circumference NS concentrick to BD .

2^o, If the Arch ME be described from the Centre O , the Part AM of the Cycloid shall be * to the Chord $AE :: 2b - 2a : b$.

3^o, The Space $MGBA$ is * to the Segment $BEZA :: 3b - 2a : b$.

4^o, If we assume $OQ = \sqrt{2aa - 3ab + bb}$, viz. a mean Proportional between OK, OA ;
K then

then the Space $ABEM$ comprehended under the Part AM of the Cycloid, the Arch ME , the Chord EB , and the Diameter AB , shall be * to the Triangle EKB :: $3b - 2a : b$. But if we make OQ or $OE = \sqrt{2aa + 2ab + bb}$, that is, the Arch AE equal to a Quadrant; the Space $AZEM$ comprehended under the Part AM of the Cycloid, and the Arches ME , AE , shall be * to the Triangle EKB , (which, in this Case, is the half of the Square of the Radius) :: $2b - 2a : b$.

* Art. 103.

* Ibid.

COROLL. V.

* FIG. 86, 105. IF the Radius OB of the immoveable
88. Circle be conceived to become infinite, then will the Arch BGD become a straight Line, and the Curve AMD will be the common Cycloid. Now since, in this Case, AB the Diameter of the moveable Circle, is nothing with respect to that of the immoveable one. Therefore, 1°, $MG : GC :: b : b$, because $b + 2a = b$, that is, $MG = GC$; and consequently, if you assume $BN = AB$, and draw the right Line NS parallel to BD , the Evolute DCN shall be generated by the Rotation of the Circle, having BN as a Diameter, on or along the Base NS .

FIG. 85, 2°, The Part AM of the Cycloid is to the
88. correspondent Chord $AE :: 2b : b$. And the Space $MGBA$ is to the Segment $BEZA :: 3b : b$.

FIG. 87, 3°, Because BQ or $\pm OQ \pm OB$, which I
88. call x , is $= \pm b \pm \sqrt{2aa + 2ab + bb}$, there comes out $xx \pm 2bx = 2aa \pm 3ab$, and $x = \frac{1}{2}a$, since all the Terms affected with b vanish, they being

ing infinitely less than the other Terms. Consequently if in the common Cycloid you assume $BP = \frac{1}{3} AB$, and draw PEM parallel to the Base $Fig. 83.$
 BD ; the Space $AMEB$ shall be the Triple of the Triangle EKB . In like manner, you will find that when P coincides with the Centre K , the Space $AZEM$ contained under the Part AM of the Cycloid, the right Line ME , and the Arch AE , shall be equal to the Square of the Radius. Which has been demonstrated before. (*Art. 99.*)

SCHOLIUM.

106. **B**ECAUSE the Arches DG, GM , are always equal to each other; the Angle DOG shall likewise be always to the Angle $GKM :: GK : OG$. Therefore when we have given D the beginning of the Cycloid DMA , the Radii OG, GK , of the generating Circles, and the Point of Contact G , if you have a mind to determine the Point M in this Position which describes the Cycloid, it is only drawing the Radius KM so, that the Angle GKM be to the given Angle $DOG :: OG : GK$. Now this may be done geometrically always, when the Relation of the said Radii can be expressed in Numbers; and so the Cycloid DMA will then be a geometrical Curve.

For let, for Example, $OG : GK :: 13 : 5$. It is manifest that the Angle MKG must be $\frac{2}{3} DOG$ the given Angle. And so the whole Matter consists in dividing the Angle DOG into five equal Parts. But every Geometrician knows, that a given Angle or Arch may be divided into any Number of equal Parts geometrically; because there always comes out

an Equation containing straight Lines only. Whence, &c.

I say, moreover, that the Cycloid DAM is a mechanical Curve; or, which is the same thing, that the Points M in it cannot be geometrically determined, when the Ratio of OG to KG can be expressed only by Surds.

FIG. 89.

For every Line, whether mechanical or geometrical, returns into itself, or else goes on (or is extended) *ad infinitum*; since the Generation thereof may be continued at pleasure. Now when the Point A , in the moveable Circle ABC , in one Revolution, has described the Cycloid ADE , this will be but the first Part of the Curve; since as the Circle rolls on, there will be described a second Part EFG , and a third GHI , &c. until the describing Point A , after several Revolutions, returns again to the same Point in the immoveable Circle from whence it went. So that if you again revolve the Circle ABC as before, the same Curve Line $ADEFGHI$ will again be described by the Point A . Now when the Radii of the generating Circles are incommensurable, their Circumferences are so likewise; and consequently the describing Point A in the moveable Circle ABC , can never return again to the Point A in the moveable Circle from whence it went, be the Number of Revolutions never so many; therefore there may be an infinite Number of Cycloids that all together make up but one Curve $ADEFGHI$, &c. Now if an indefinite right Line be drawn thro' the immoveable Circle, it is plain that it shall cut the Curve continued *ad infinitum* in an infinite Number of Points. But because the Equation expressing the Nature of a geometrical

metrical Line, must at least be of as many Dimensions as is the Number of Points that a straight Line can cut the Curve in. Therefore the Equation that expresses the Nature of the Curve $ADEFGHI$, &c. must be of an infinite Number of Dimensions: Which being impossible, it is plain that the said Curve is a mechanical or transcendent one.

PROP. III.

107. **T**O find any Number of Involutés AM , BN , EFO , to the given Evolute BFC . FIG. 90.

It is manifest that the Points A, B, F , of the Thread $ABFC$ of the Evolute BFC , will describe the Curves AM, BN, FO , to which the given Curve BFC is the common Evolute. But because the Curve FO is described by the Evolution of the Part FC , it does not begin in F . And in order to find where it does begin, the Part BF remaining, must be taken as the Evolute, the Point B being that to which the Thread is fixed; and beginning at F , the Part EF of the Involute EFO must be described, which begins in E , and is the Involute to the whole Curve BFC .

If you have a mind to find the Points M, N, O , without the Thread $ABFC$, you need only assume the Parts CM, CN, CO , equal to $ABFC, BFC, FC$, in any Tangent CM excepting BA .

COROL.

108. **H**ENCE it follows, 1^o, That the involute Curves AM, BN, EFO , differ very much from each other. I mean as to their Na-

ture; since in the Vertex A of the Curve AM , the Radius of Evolution is equal to AB , whereas that of the Curve BN is nothing. It is likewise evident from the very Figure of the Curve EFO , that it differs very much from the Curves AM, BN .

2^o, That the Curves AM, BN, EFO , are geometrical ones only, when the given Evolute BFC , is a geometrical Curve, and rectifiable: For if the same be not a geometrical Curve, when BK is assumed for an Absciss, the Ordinate KC cannot be determined geometrically; And if it be not rectifiable, when the Tangent CM is drawn, the Points M, N, O , cannot be determined in the Curves AM, BN, EFO , because straight Lines cannot be found equal to the Curve BFC , and the Parts BF, FC thereof.

SCHOLIUM.

FIG. 91. 109. **I**F the Evolute BAC has a Point of Inflection in A ; then from the Evolution of the Part BAD thereof, beginning at the Point D (not the Point of Inflection) there will be formed the Part DEF of the Involute, and from the Evolution of the Part DC , will be generated the remaining Part DG of the Involute: So that the whole involute Curve formed from the Evolution of the Curve BAC will be $FEDG$. Now it is evident that this Curve has two Points D and E of Retrogression, with this Difference, that at D the Parts DE, DG , have opposite Convexities; and at E , the Concavities of DE, EF , lie the same way. Now in the Section foregoing, the Determination of Points of Retrogression

of the same Nature with D was handled, therefore I shall here shew the Manner of finding the Points E , which may be called Points of Retrogression of the second Kind; and this is what no body, that I know of, has hitherto consider'd.

In order to which, draw at pleasure two Perpendiculars MN, mn , to the Part DE terminating in the Points N, n , of the Evolute; from which draw two other Perpendiculars NH, nH to NM, nm ; then the small Sectors MNm, NHn , will be similar, because the Angles MNn, NHn , are equal. Therefore $Nn : Mm :: NH : NM$. Now in the Point A of Inflexion, the Radius NH becomes * infinite * Art. 81. or nothing; and the Radius MN , which becomes AE , continues finite. Therefore in the Point E of Retrogression of the second Kind, the Ratio of Nn , the Fluxion of the Radius of Evolution MN , to Mm the Fluxion of the Curve, must become infinitely great or or infinitely small. And because * $Nn =$ * Art. 86.

$$\frac{-3\dot{x}\dot{y}\ddot{y}^2\dot{x}^2 + \dot{y}^2 + \dot{x}\ddot{y}\dot{x}^2 + \dot{y}^2}{\dot{x}^2\dot{y}^2}, \text{ and } Mm = \sqrt{\dot{x}^2 + \dot{y}^2}, \text{ therefore } \frac{\dot{x}^2\dot{y} + \dot{y}^2\dot{y} - 3\dot{y}\ddot{y}^2}{\dot{x}\dot{y}^2} = 0, \text{ or In-}$$

finiteness; and multiplying by $\dot{x}\dot{y}^2$, we have the following general Expression $\dot{x}^2\dot{y} + \dot{y}^2\dot{y} - 3\dot{y}\ddot{y}^2 = 0$, or Infinity, for determining the Points of Retrogression of the second Kind.

If an Involute DEF , or $HDEFG$, has a Point of Retrogression of the second Kind, the Evolute BAC may have a Point of Retrogression of the second Kind also; so that the second Point A of Retrogression answers to the second Point E , viz. both lie in the Radius of

Evolution issuing from the Point *E*. Now in this Supposition it is evident, that the Radius of Evolution *E A*, will be always a *Minimum* or *Maximum*. And therefore the Fluxion of

• Art. 78. $\frac{x^2 + y^2}{x^2}$ the general Expression * for the Ra-

dii of Evolution, must be 0, or infinite in the Point *E* sought; from whence we get the same general Expression as before; which must be used to investigate the Points of Retrogression of the second Kind.



S E C T.



SECT. VI.

*The Use of Fluxions in finding of
Causticks by Reflexion.*

DEFINITION.

IF an infinite Number of Rays $BA, BM, BD,$ FIG. 94,
95.
issuing from a luminous Point B , be reflected by the Curve AMD in such manner, that the Angles of Reflexion be equal to the Angles of Incidence; the Line HFN that touches the reflected Rays, or their Continuations AH, MF, DN , is called a *Caustick by Reflexion*.

COROL. I.

110. **I**F HA be continued out to I , so that FIG. 94.
 $AI = AB$, and the Caustick HFN be taken as an Evolute, and IA as the first Radius of Evolution, then will the Involute ILK to the same, be of such a Nature, that the Tangent FL shall be * constantly equal to the * Art. 73.
Part FH of the Caustick Plus the right Line HI . And if Bm, mF , be supposed two reflected Rays infinitely near BM, MF ; and if Fm be continued out to l , and the little Arches MO, MR , be described from the Centres F, B ; the small right-angled Triangles MOm, MRm , will be equal and similar;
Since

Since the Angle $O m M = F m D = R m M$, and the Hypothenufe $M m$ is common; so that $O m = R m$. Now because $O m$ is the Fluxion of $L M$, and $R m$ the Fluxion of $B M$, and this is always so, wherever the Point M be taken: Therefore $M L - I A$, or $A H + H F - M F$,

- * Art. 96. the Sum of all * the Fluxions $O m$ in the Part $A M$ of the Curve, is $= B M - B A$, the Sum
 * Art. 96. * of all the Fluxions $R m$ in the same Part $A M$; and consequently the Part $H F$ of the Caustick $H F N$, will be equal to $B M - B A + M F - A H$.

There may happen several Cases, according as the incident Ray $B A$ is greater or less than $B M$, and the reflected Ray $H A$, as a Radius of Evolution disengages itself, from the Part $H F$ of the Curve, to become $M F$: But we can always prove, as we have already, that the Difference of the Radii of Incidence, is equal to the Difference of the reflected Rays Plus the Part of the Caustick taken as an Evolute that one of the Rays disengages itself from, before it coincides with the other. For Example, $B M - B A = M F + F H - A H$; and consequently $F H = B M - B A + A H - M F$.

FIG. 95.

FIG. 94.

95.

FIG. 96.

If the Arch $A P$ be described from the Centre B , it is plain that $P M$ shall be the Difference of the incident Rays $B M, B A$. And if the luminous Point B becomes infinitely distant from the Curve $A M D$, the incident Rays $B A, B M$, will become parallel, and the Arch $A P$ a straight Line perpendicular to those Rays.

COROL. II.

FIG. 94.

III. If the Figure $B A M D$ be supposed to be inverted on the same Plane, so that the Point

Point B coincides with I , and the Tangent to the Curve AMD in its first Situation, still touches it in this latter one; and if the Curve aMd revolves on AMD , viz. on itself, so that the Parts aM , AM , be always equal. I say, by this Motion the Point B will describe a kind of Cycloid ILK , which will be an Involute to the Caustick HFN , taken as an Evolute.

For from the Generation it follows, 1° That the Line LM drawn from the describent Point L , to the Point of Contact M , will be * per- * Art. 43. pendicular to the Curve ILK . 2°, That La or $IA=B A$, and $LM=BM$. 3°, That the Angles made by the right Lines ML , BM , with the common Tangent in M are equal. And therefore if LM be continued out to F , MF shall be the Incident Ray BM reflected. Whence the Perpendicular LF touches the Caustick HFN ; and since this is so always, let the Point L be taken where it will, it is plain that ILK is the Involute to the Caustick HFN , the right Line HI being the Radius of Evolution.

Hence it follows, that the Part FH or $FL - HI = BM + MF - BA - AH$. Which is what has been otherwise demonstrated in the Corollary foregoing.

COROL. III.

112. **W**HEN the Tangent DN is infinitely near the Tangent FM , it is manifest that the Point of Contact N , and the Point V of Intersection, will both coincide with F the other Point of Contact: So that the Point F , wherein the reflected Ray MF touches

touches the Caustick HFN is determin'd, with only seeking the Concurrence of the infinitely near reflected Rays MF, mF . Consequently if we suppose an infinite Number of infinitely near incident Rays, the Intersections of these Rays reflected, will form a Polygon of an infinite Number of Sides, viz. the Caustick HFN .

PROP. I.

FIG. 97. 113. **T**HE Nature of the Curve AMD , the luminous Point B , and the incident Ray BM being given: To find the Point F in the reflected Ray MF given in Position, wherein it touches the Caustick.

FIND (by Section foregoing) MC the Length of the Radius of Evolution in the Point M , assume the infinitely small Arch Mm , draw the right Lines Bm, Cm, Fm , from the Centres B, F , describe the small Arches MR, MO , draw CE, Ce, CG, Cg , perpendicular to the incident and reflected Rays; and lastly, call the given Quantities BM, y ; ME or MG, a .

*Art. 110. Now we prove (as in *Coroll. 1.) that the Triangles MRm, MOm , are equal and similar; and so $MR=MO$. But because of the Equality of the Angles of Incidence and Reflexion, we have also $CE=CG, Ce=Cg$; and therefore $CE-Ce$ or $E\mathcal{Q}=CG-Cg$ or SG . Whence because of the similar Triangles $BMR, BE\mathcal{Q}$; FMO, FGS , $BM+BE(2y-a):BM(y)::MR+E\mathcal{Q}$ or $MO+GS:MR$ or $MO::MG(a):MF=\frac{ay}{2a-y}$.

If the luminous Point B falls on the other Side the Point E , with respect to the Point M ,

or (which is the same thing) if the Curve *AMD* be convex next to the luminous Point *B*; then will *y* be negative, and consequently $MF =$

$$\frac{-ay}{-2y-a} \text{ or } \frac{ay}{2y+a}$$

If *y* be infinite, that is, if the Point *B* be at an infinite Distance from the Curve *AMD*, then the incident Rays will be parallel, and *MF* will be $= \frac{1}{2}a$, because *a* is nothing with respect to *2y*.

COROL. I.

114. **B**ECAUSE *MF* has but one Value, wherein is the Radius of Evolution;

therefore the Curve *AMD* can have only one Caustick by Reflexion *HFN*, since there is but one Evolute to it.

COROL. II.

115. **W**HEN the Involute *AMD* is a geometrical Curve, it is manifest the Evolute is one likewise; that is, any Point *C* in it may be determined geometrically; and consequently any Point *F* in the Caustick thereof, may be determined geometrically, or (which is the same) the Caustick *HFN* will be a geometrical Curve. I say, moreover, that the said Caustick is always rectifiable; because by means of the Curve *AMD*, which is supposed to be a geometrical one, it is manifest that straight Lines may be found equal to any Part thereof.

COROL.

C O R O L. III.

FIG. 97. 116. IF the Curve AMD be convex next to the luminous Point B ; the Value of

$MF \left(\frac{ay}{2y+a} \right)$ will always be positive; and

the Point F must of consequence be assumed on the same Side the Point M as the Point C , as we have supposed in the Investigation foregoing. Therefore the infinitely-near reflected Rays do diverge.

But if the Curve AMD be concave next to the luminous Point B ; $MF \left(\frac{ay}{2y-a} \right)$ will

be positive, when y is greater than $\frac{1}{2}a$, negative when the same is less, and infinite when it is $= \frac{1}{2}a$. Whence if a Circle be described with a Diameter equal to $\frac{1}{2}MC$ the Radius of Evolution, the infinitely-near reflected Rays will converge, when the luminous Point B falls without the Circumference thereof, and diverge when the same falls within; and finally will be parallel when it falls in the Circumference.

C O R O L. IV.

117. IF the incident Ray BM touches the Curve AMD in the Point M , then will $ME(a) = 0$; and therefore $MF = 0$. But because the reflected Ray is then in the Direction of the incident Ray, and it is the Nature of the Caustick to touch all the reflected Rays; therefore it shall touch likewise the incident Ray BM in the Point M ; that is,

is, the Caustick and the given Curve will have the same Tangent in the common Point M .

If the Radius CM of Evolution be nothing, then still will $ME(a) = 0$; and consequently $MF = 0$. Therefore the given Curve, and the Caustick, make an Angle with each other in the common Point M , equal to the Angle of Incidence.

If the Radius of Evolution CM be infinite, the small Arch Mm will become a straight Line, and $MF = -y$; because $ME(a)$ being infinite, y will be nothing with regard to a . Now since this Expression or Value is negative when the Points B, C , are both on the same Side the Line AMD , and positive when one is on one side, and the other on the other. Therefore the infinitely near reflected Rays will always diverge when AMD is a right Line.

C O R O L L. V.

118. **I**T is evident, that when any two of the Points B, C, F , are given, the third is easily found.

1° Let the involute Curve AMD be a Parabola, and the luminous Point B the Focus. Then (from the Conick Elements) it is manifest, that all the reflected Rays will be parallel to the Axis; and so in every Position of the Point M , MF will be infinite; and consequently $a = 2y$. Whence if you assume $ME = 2MB$, and draw the Perpendicular EC , this will intersect the Perpendicular MC to the Curve AMD , in the Point C , which will be in the Evolute.

FIG. 98.

2° Let

FIG. 99. 2^o, Let AMD be an Ellipsis, and the luminous Point B one of the Foci. Then it is evident, that all the reflected Rays MF will coincide in the other Focus F . And if you call

*Art. 113. MF, z ; then will $*z = \frac{ay}{2y-a}$; from whence comes out $ME(a) = \frac{2yz}{y+z}$. Which was sought.

FIG. 110. But if the Curve AMD be an Hyperbola, the Focus F will fall the contrary way, or without the Curve; and therefore $MF(z)$ will become negative; and consequently $ME(a) = \frac{-2yz}{y-z}$ or $\frac{2yz}{z-y}$. Whence arises the following Construction, which will serve for the Ellipsis also.

FIG. 99, 100. Assume ME a fourth Proportional to half the transverse Axis, and the incident and reflected Rays, and draw the Perpendicular EC : This shall intersect the Line MC perpendicular to the Curve in the Point C , which shall be in the Evolute.

EXAMPLE I.

FIG. 101. 119. LET AMD be a Parabola, and let the incident Rays PM fall perpendicular to the Axis. It is required to find the Points F in the reflected Rays MF , wherein they touch the Caustick AFK .

*Art. 113. If we draw MC the Radius of Evolution, and the Line CG perpendicular to the reflected Ray MF , it is manifest * that we must assume $MF = \frac{1}{2}MG$. But this Construction may be shorten'd, if you consider that when MN is drawn parallel to the Axis, AP , and the

the right Line ML to the Focus L , the Angles LMP , FMN shall be equal, since from the Nature of the Parabola $LMQ = QMN$, and by the Supposition $PMQ = QMF$. If then the common Angle PMF be added to both, the Angle LMF shall be equal to the Angle PMN , that is, a right Angle. Now it has been demonstrated (*Art. 118. Num. 1.*) that LH perpendicular to ML shall intersect the Radius of Evolution MC in H the middle thereof. Therefore if MF be drawn equal and parallel to LH , it will be one of the reflected Rays, and will touch the Caustick AFK in the Point F . Which was to be found.

If the reflected Ray MF be supposed parallel to the Axis AP , it is manifest that the Point F of the Caustick will be at the greatest Distance possible from AP , because the Tangent to that Point will be parallel to the Axis. And consequently in order to determine that Point in all the Causticks, as AFK , formed by incident Rays parallel to the Axis of the given Curve, we need only consider that then MQ must be equal to PQ . And therefore $y = x$.

Now let $ax = yy$. Then $y = \frac{a\dot{x}}{2\sqrt{ax}} = \dot{x}$,

and so $AP(x) = \frac{1}{4}a$: that is, when the Point P falls in the Focus L , the reflected Ray MF will be parallel to the Axis; which is a known Truth otherwise establish'd; since in this Case MP coinciding with LM , MF must also coincide with MN , and LH with LQ . Whence MF will then be equal to ML ; and therefore if FR be drawn perpendicular to the Axis, AR or $AL + ME$ will be $= \frac{1}{4}a$. We may observe,

serve, that in this Case the Part AF of the Caustick is equal to the Parameter, since it is

* Art. 110. * always equal to $PM + MF$.

To determine the Point K wherein the Caustick AFK intersects the Axis AP , the Value of MO must be found, and made equal to that of MF ; for it is plain, when the Point F falls in K , the Lines MF, MO become equal to each other. For when the Point F coincides with K , it is manifest that the Lines MF, MO will become equal to one another. Therefore if the unknown Quantity MO be called t ; from the Bisection of the Angle PMO by MQ perpendicular to the Curve, we shall have $MP(y)$:

$$MO(t) :: P Q \left(\frac{y \dot{y}}{\dot{x}} \right) : O Q = \frac{t \dot{y}}{\dot{x}}. \text{ And}$$

$$\text{therefore } OP = \frac{t \dot{y} + y \dot{y}}{\dot{x}} = \sqrt{tt - yy}, \text{ because}$$

of the right-angled Triangle MPO ; and dividing both Sides by $t + y$, there comes out

$$\frac{\dot{y}}{\dot{x}} = \sqrt{\frac{t - y}{t + y}}, \text{ from whence we get } MO(t) =$$

$$* \text{ Art. 77. } \frac{y \dot{x}^2 + y \dot{y}^2}{\dot{x}^2 - \dot{y}^2} = MF(\frac{1}{2}a) = \frac{\dot{x}^2 + \dot{y}^2}{-2\ddot{y}}, \text{ since } * ME$$

$$(a) = \frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}}. \text{ And so by means of } \dot{y}^2 - 2y\ddot{y}$$

$+ \dot{x}^2$, the Point P may be determined so, that drawing the incident Ray PM , and the reflected Ray MF , this latter will touch the Caustick AFK in the Point K , wherein it intersects the Axis AP .

Now in the Parabola $y = x^{\frac{1}{2}}$, $\dot{y} = \frac{1}{2}x^{-\frac{1}{2}}$,
 $\ddot{y} = -\frac{1}{4}x^{-\frac{3}{2}}$, and substituting these Expressions in the foregoing Equation, there will
 come

come out $\frac{1}{4}x^{-1}\dot{x}^2 + \frac{1}{2}x^{-1}\dot{x}^2 = \dot{x}^2$; and so we shall get $AP(x) = \frac{1}{4}$ of the Parameter.

Now to find the Nature of the Caustick AFK after the manner of *Descartes*, we must get an Equation expressing the Relation of the Absciss $AR(u)$ to the Ordinate $RF(z)$; which may be done thus. Because $MO(t) = \frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 - \dot{y}^2}$, therefore $PO\left(\frac{t\dot{y} + y\dot{y}}{\dot{x}}\right) = \frac{2y\dot{x}\dot{y}}{\dot{x}^2 - \dot{y}^2}$;

and because of the similar Triangles MPO ,

MSF , therefore $MO\left(\frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 - \dot{y}^2}\right) : MF$

$\left(\frac{\dot{x}^2 + \dot{y}^2}{-2\dot{y}}\right)$ or $-2y\dot{y} : \dot{x}^2 - \dot{y}^2 :: MP(y) : MS$

$(y - z) = \frac{\dot{x}^2 - \dot{y}^2}{-2\dot{y}} :: PO\left(\frac{2y\dot{x}\dot{y}}{\dot{x}^2 - \dot{y}^2}\right) : SF$ or PR

$(u - x) = \frac{\dot{x}\dot{y}}{-\dot{y}}$. Therefore with $z = y +$

$\frac{\dot{y}^2 - \dot{x}^2}{-2\dot{y}}$, and $u = x + \frac{\dot{x}\dot{y}}{-\dot{y}}$ together with the

Equation of the given Curve, we get a new Equation freed from x and y , which expresses the Relation of $AR(u)$ to $FR(z)$.

When the Curve AMD is a Parabola, as in the Example, we shall get $z = \frac{1}{2}x^{\frac{1}{2}} - 2x^{\frac{3}{2}}$, or (squaring each Side) $\frac{1}{4}x - 6xx + 4x^3 = zz$, and $u = 3x$. Whence we get the Equation sought $az = \frac{1}{4}u^2 - \frac{2}{3}auu + \frac{1}{4}aau$ expressing the Nature of the Caustick AFK . Here we may observe, that PR is always the Double of AP , because $AR(u) = 3x$. From whence we get moreover another way of determining the sought Point F in the reflected Ray MF .

EXAMPLE II.

FIG. 101. 120. **L**ET AMD be a Semicircle, the Line AD a Diameter, and C the Centre, and let the incident Rays PM be perpendicular to AD .

Because the Evolute of the Circle is a Point, viz. the Centre of it, therefore if CM be bisected in H , and HF be drawn perpendicular to the reflected Ray MF , it shall * cut the said Ray in the Point F wherein it touches the Caustick AFK . Now because the reflected Ray MF is equal to $\frac{1}{2}$ of the incident Ray PM ; therefore when the Point P coincides with C , it is plain the Point F will coincide with K the middle of CB . And the Part AF is the triple of MF , and the Caustick AFK the triple of BK . We may likewise observe, that when the Angle ACM is made one half a right Angle, the reflected Ray MF shall be parallel to AC ; and therefore the Point F will be the highest Point of the Caustick above the Diameter AD .

The Circle whose Diameter is the Line MH passes thro' the Point F ; because the Angle HFM is a right Angle. And if from the Centre C , with the Radius CK or CH the half of CM , the Circle KHG be described; the Arch HF shall be equal to the Arch HK : for since the Angle CMF is equal to CMP or HCK , the Arches $\frac{1}{2} HF$ and HK that measure the said Angles in the Circles MFH , KHG shall be to each other as the Radii $\frac{1}{2} MH$, HC of these Circles. Whence it appears, that the Caustick AFK is a Cycloid generated by the Rotation of the Circle

cle MFH along the immoveable Circle KHG , the Beginning thereof being K , and the Vertex A .

EXAMPLE III.

121. **L**ET AMD be a Circle, the Line AD FIG. 103. the Diameter, and the Point C the Centre; and let A , one End of that Diameter, be the luminous Point from whence all the Rays of Incidence AM issue.

From the Centre C draw CE perpendicular to the incident Ray AM ; then from the Nature of the Circle, it is manifest that the Point E bisects the Chord AM ; and so $ME (a) = \frac{1}{2}y$. Whence $MF \left(\frac{ay}{2y-a} \right) = \frac{1}{2}y$: that is, we must take the reflected Ray $MF = \frac{1}{2}AM$ the incident Ray. Consequently $DK = \frac{1}{2}AD$, $CK = \frac{1}{2}CD$, and the * Caustick $AFK = \frac{1}{2}AD$, like *Art. 110. as its Part $AF = \frac{1}{2}AM$. If you assume $AM = AC$, the reflected Ray MF will be parallel to the Diameter AD ; and consequently the Point F will be the highest Point possible above the said Diameter.

If you take $CH = \frac{1}{2}CM$, and draw HF perpendicular to MF ; the Point F shall be in the Caustick: for drawing HL perpendicular to AM , it is manifest that $ML = \frac{1}{2}ME = \frac{1}{4}AM$, because $MH = \frac{1}{2}CM$. Therefore the Circle having MH as a Diameter, shall pass thro' the Point F of the Caustick; and if from the Centre C with the Radius CK or CH another Circle KHG be described, it shall be equal to the former one, and the Arch HK shall be equal to the Arch HF : For in the Isosceles Triangle CMA the external Angle

$KCH = 2CMA = AMF$; and therefore the Arches HK , HF being the Measures of those Angles in the equal Circles, shall likewise be equal. Whence it follows, that the Caustick AFK is likewise here a Cycloid generated by the Rotation of the Circle MFH along the immoveable one KHG , the Beginning being K , and the Vertex A .

This may be demonstrated otherwise thus. If a Cycloid be described by the Revolution of a Circle equal to the Circle AMD about this same, beginning at A , we have demonstrated (*Art.* 111.) that the Caustick AFK will be the Evolute to the said Cycloid taken as an Involute. But *the said Evolute is a Cycloid of the same Kind, viz. the Diameters of the generating Circles of them shall be equal; and the Point K will be determined by taking CK a third Proportional to $CD + DA$ and CD , viz. equal to $\frac{1}{3} CD$. Whence, &c.

EXAMPLE IV.

FIG. 104. 122. **L**ET the Curve AMD be one half of a common Cycloid described by the Rotation of the Semicircle NCM along the right Line BD , whose Vertex is A , and the Beginning thereof D ; and let the incident Rays KM be parallel to the Axis AB .

FIG. 104. Because * MG is equal to the half of the Radius of Evolution; therefore * if GF be drawn perpendicular to the reflected Ray MF , the Point F shall be in the Caustick DFB . And so MF must be assumed equal to KM .

* *Art.* 95.

* *Art.* 113.

If

If from H the Centre of the generating Circle MGN to the Point of Contact G , and to the describent Point M be drawn the Radii HG, HM ; it is evident that HG will be perpendicular to BD , and the Angle $GMH = MGH = GMK$: whence the reflected Ray MF passes thro' the Centre H . Now the Circle having GH as a Diameter, passes thro' the Point F likewise: because the Angle GFH is a right Angle. Therefore the Arches $GN, \frac{1}{2} GF$ the Measures of the same Angle GHN shall be to one another, as the Diameters MN, GH of their Circles. And consequently the Arch $GF = GN = GB$. Whence it is evident, that the Caustick DFB is a Cycloid described by the entire Rotation of the Circle GFH along the right Line BD .

EXAMPLE V.

123. **L**ET the Curve AMD be still the half FIG. 105. of a common Cycloid, the Base BD whereof is equal to one half the Circumference ANB of the generating Circle; and let the incident Rays PM be parallel to the Base BD .

If GQ be drawn perpendicular to PM , the right-angled Triangles GQM, BPN will be equal and similar; and therefore $MQ = PN$. Whence it follows, * that MF must be assu- * *Art. 95,* med equal to the correspondent Ordinate PN ^{113.} in the generating Circle ANB .

When the Point F is at the greatest Distance possible from the Axis AB , the Tangent MF in that Point must be parallel to the Axis. And consequently the Angle PMF will be a right Angle, and PMG or PNB half a right

Angle. Whence the Point P falls in the Centre of the Circle AND .

Here it is worth observing, that as the Point P afterwards accedes towards the Extremity B of the Diameter, so does the Point F likewise accede towards the Axis AB , until it comes to a certain Point K ; after which it recedes therefrom till it comes to D . So that the Caustick $AFKFD$ has a Point of Retrogression in K .

*Art. 110,
III.

To determine which, we must observe* that the Part $AF = PM + MF$, the Part $AFK = HL + LK$, and the Part KF of KFD is $= HL + LK - PM - MF$: whence $HL + LK$ must be a *maximum*. And if you call AH, x ; HI, y ; the Arch AI, u ; then will $HL + LK = u + 2y$, the Fluxion whereof is $\dot{u} + 2\dot{y} = 0$, and $\frac{ax}{y} + 2\dot{y} = 0$, by substitu-

ting $\frac{ax}{y}$ for $\dot{u}$: from whence we get $a\dot{x} = -2y\dot{y} = 2x\dot{x} - 2a\dot{x}$ because of the Circle: and therefore $AH(x) = \frac{1}{2}a$.

COROL.

124. **T**HE Space AFM or $AFKFM$ contained under the Parts of the Curves AF or $AFKE$, AM , and the reflected Ray MF is equal to the half of the circular Space APN . For the Fluxion thereof, *viz.* the Sector FMO is equal to the half of the Rectangle $PpSN$, the Fluxion of the Space APN ; since the right-angled Triangles MOm , MRm being similar and equal, MO shall be equal to MR or NS or Pp , and moreover $MF = PN$.

EXAM-

EXAMPLE VI.

125. **L**ET the Curve AMD be the half of FIG. 106.

a Cycloid generated by the Rotation of the Circle MGN about AGK equal to it, and let A be the beginning thereof, and D the Vertex. And let the incident Rays AM all issue from the Point A . The Line BH joining the Centres of the generating Circles constantly passes thro' the Point of Contact G , and the Arches GM, GA , as also their Chords, are always equal; also the Angle $HGM = BGA$, and the Angle $GMA = GAM$. Now the Angle $HGM + BGA = GMA + GAM$; because if the Angle AGM be added to each Side, there will be two right Angles made. Whence the Angle HGM shall be always equal to the Angle GMA ; and so likewise to the Angle of Reflection GMF . Whence it follows, that MF always passes thro' H the Centre of the moveable Circle.

Now if CE, GO , be drawn perpendicular to the incident Ray AM , it is manifest that $MO = OA$, and $OE = \frac{1}{3}OM$; since the Point C being* in the Evolute, $GC = \frac{1}{3}GM$; there- *Art. 100.
fore $ME = \frac{1}{3}AM$; that is, $a = \frac{2}{3}y$; and con-

sequently $MF \left(\frac{ay}{2y-a} \right) = \frac{1}{3}y$. Whence if you draw GF perpendicular to MF , the Point F will be in the Caustick AFK .

The Circle whereof GH is a Diameter, does pass thro' the Point F ; and since the Arches $GM, \frac{1}{2}GF$, the Measures of the Angle GHM are to one another, as the Diameters MN, GH , of their Circles; the Arch GF shall be equal to GM , and consequently to the Arch GA .

From

From whence it is evident, that the Caustick AFK is a Cycloid generated by the Rotation of the moveable Circle HFG about or along the immoveable one AGK .

COROLLARY.

126. **I**F a Circle be described about the Centre B , with a Radius equal to BH or AK ; and an infinite Number of right Lines parallel to BD falls on the Circumference of it: it is then manifest * that by their Reflection they will form the same Caustick AFK .

* Art. 120.

EXAMPLE VII.

FIG. 107. 127. **L**ET AMD be a Logarithmick Spiral, and let the incident Rays (AM) all issue from the Centre A .

If the right Line CA be drawn from C , the End of the Radius of Evolution, perpendicular to the incident Ray AM , it shall meet * the same in the Centre A . Therefore AM

* Art. 91.

$(y) = a$; and consequently $MF \left(\frac{ay}{2y-a} \right) = y$. Whence AMF shall be an Isosceles Triangle; and since the Angles of Incidence and Reflection AMT , FMS , are equal to one another, the Angle AFM is equal to the Angle AMT . Whence it is manifest that the Caustick AFK will be a Logarithmick Spiral, differing only from the proposed one AMD in Position.

PROP.

PROP. II.

128. **T**HE Caustick HF by Reflection, and the luminous Point B being given, to find an infinite Number of Curves, as AM , whereof the same is the Caustick by Reflection. FIG. 108.

Take the Point A at pleasure in any Tangent HA , for one of the Points of the Curve AM sought; and from the Centre B , with the Distance BA , describe the circular Arch AP ; and with any other Distance BM , another circular Arch. Then assume $AH + HE = BM - BA$ or PM ; and by the Evolution of the Caustick HF , beginning at E , describe the involute Curve, cutting the circular Arch described with the Radius BM in the Point M , which will be * in the Curve AM . For by *Art. 110. Construction $PM + MF = AH + HF$.

Or else take a Thread BMF , and having fixed one End in B , and the other in F , stretch the Thread by Means of a Pin at M , which so move along, that the Part MF of the Thread wraps about the Caustick HF : Then it is evident that the Pin M in thus moving, describes the Curve MA sought.

Otherwise thus:

129. **D**RAW any Tangent (FM) excepting HA , at pleasure, and in the same find the Point M such, that $BM + MF = BA + AH + HF$; which may be done thus:

Assume $FK = BA + AH + HF$, and bisecting BK in G , draw the Perpendicular GM .

This

This shall intersect the Tangent FM in the sought Point M : for $BM = MK$.

FIG. 109.

If the Point B be supposed at an infinite Distance from the Curve AM ; that is, if the incident Rays BA, BM , be parallel to a right Line given in Position; the former Construction will serve likewise in this Case, in conceiving the Arches described from the Centre B , to become right Lines perpendicular to the incident Ray. But the latter Construction will not do here; for which we shall lay down the following one.

Assume $FK = AH + HF$. And find the Point M such, that MP (a Parallel to AB) perpendicular to AP , be equal to MK ; then

*Art. 110. it is plain * that M will be the Point sought in the Curve AM , because $PM + MF = AH + HF$. Now this is done thus.

Draw KG perpendicular to AP ; and assuming $KO = KG$, draw KP parallel to OG , and PN parallel to GK : I say the Point M will be that required. For because of the similar Triangles GKO, PMK , PM will be $= MK$; since $GK = KO$. If the Caustick HF degenerates into a Point, the Curve AM will become a Conick Section.

COROL. I.

130. **H**ENCE it is manifest, that the Curve, which passes thro' all the Points K , is generated by the Evolution of the Curve HF , beginning at A , and that its Nature varies according as the Situation of the Point A in the Tangent AH varies. Wherefore because the Curves (AM) are all generated by the same geometrical Construction from the said

said Curves: It is manifest * that the Nature * *Art. 108.*
of them differ, and they are only geometrical
when the Caustick *HF* is a geometrical Curve
and rectifiable.

COROL. II.

131. A CURVE *DN*, together with the lu- FIG. 110.
minous Point *C* being given, to find
any Number of Lines (as *AM*) being such,
that they may cause all the Rays *DA*, *NM*,
reflected from them, to unite in a given Point
B.

If the Curve *HF* be supposed to be the Cau-
stick of the given Curve *DN*, formed by the
luminous Point *C*; it is manifest that the said
Line *HF* must likewise be the Caustick of the
Curve *AM*, having the given Point *B* as a
luminous Point: So that $FK = BA + AH +$
 HF , and $NK = BA + AH + HF + FN = BA$
 $+ AD + DC - CN$, since * $HD + DC = HF$ * *Art. 110.*
 $+ FN + NC$. Which gives this Construc-
tion.

In any reflected Ray assume the Point *A* at
pleasure, for one Point in the sought Curve
AM; and in any other reflected Ray *MN*, as-
sume the Part $NK = BA + AD + DC - CN$,
and the Point *M* will be found as above (*Art.*
129.)





S E C T. VII.

*The Use of Fluxions in finding of
Causticks by Refraction.*

D E F I N I T I O N.

- FIG. III. **I**F the Directions of an infinite Number of Rays BA, BM, BD , all issuing from the same luminous Point B , be alter'd after their Concurrence with a Curve AMD , either nearer or farther from the Perpendiculars MC to it: And if the Law of Alteration be constantly such, that (CE) the Sine of the Angles of Incidence (CME) be to (CG) the Sines of the Angles (CMG) of Refraction, in the given
- FIG. 112. Ratio of m to n ; the Curve HFN that touches all the broken or refracted Rays, or the Continuations of them, AH, MF, DN , is called the *Caustick by Refraction*.

C O R O L L A R Y.

132. **I**F the Caustick HFN be taken as an Evolute, and the Involute ALK be described from it beginning at A ; the Line LF Plus FH , a Part of the Caustick, will be always equal to AH . And if you conceive another Tangent Fml infinitely near to FML , and another Ray of Incidence Bm ; and from the Centres F, B , be described the small Arches MO ,

MO, MR ; then the little right-angled Triangles MRm, MOm , will be similar, the former to $M E C$, and the latter to $M G C$; because if the Angle EMm be taken from the right Angles RME, CMm , the Angles RMm, EMC , remaining, shall be equal: and, in like manner, if the Angle GMm be taken away from the right Angles GMO, CMm , the remaining AMm, GMC , will be equal. Therefore $Rm : Om :: CE : CG :: m : n$. Now since Rm is the Fluxion of BM , and Om the Fluxion of LM ; $BM - BA$ * the Sum of all the Fluxions (Rm) in the Part of the Curve AM , will be * to ML or $AH - MF - FH$, the Sum of all the Fluxions (Om) in the same Part, as m to n . And therefore the Part $FH = AH - MF + \frac{n}{m} BA - \frac{n}{m} BM$. * Art. 96.

There are several Cases, according as the incident Ray BA is greater or less than BM , and the refracted Ray AH wraps about, or unwraps (itself from) the Part HF : But we prove, as already, that the Difference between the incident Rays, is always to the Difference of the refracted Rays, (Plus the Part of the Caustick that one of these Rays disengages itself from before it falls on the other) as m to n . For Example, $BA - BM : AH - MF - FH :: m : n$. From whence we get $FH = AH - MF + \frac{n}{m} BM - \frac{n}{m} BA$. FIG. 112.

If the Arch AP be described from the Centre B , it is manifest that PM will be the Difference of the incident Rays BM, BA . And if the luminous Point B becomes infinitely distant from the Curve AMD , the incident Rays BA, BM , will be parallel, and the Arch AP FIG. 113.

AP will become a right Line perpendicular to the said Rays.

PROP. I.

FIG. III. 133. **T**HE Nature of the Curve *AMD*, the luminous Point *B*, and the Ray of Incidence *BM*, being given, to find the Point *F* in the refracted Ray *MF* given in Position, wherein the same touches the Caustick by Refraction.

*SECT. V. **F**IRST, find * the Length *MC* of the Radius of Evolution at the given Point *M*, assume the infinitely small Arch *Mm*, draw the right Lines *Bm*, *Cm*, *Fm*, from the Centres *B*, *F*, describe the little Arches *MR*, *MO*, draw *CE*, *Ce*, *CG*, *Cg*, perpendicular to the incident and refracted Rays; and call the given Quantities *BM*, *y*; *ME*, *a*; *MG*, *b*; and the little Arch *MR*, $\dot{x}$. This done,

Because of the right-angled similar Triangles *MEC* and *MRm*, *MGC* and *MOm*, *BMR* and *BQe*, $ME (a) : MG (b) :: MR (\dot{x}) : MO = \frac{bx}{a}$. And $BM (y) : BQ \text{ or } BE (y+a) ::$

$MR (\dot{x}) : Qe = \frac{a\dot{x} + y\dot{x}}{y}$. Now from the Nature of Refraction $Ce : Cg :: CE : CG :: m : n$.

And therefore $m : n :: Ce - CE \text{ or } Qe \left(\frac{a\dot{x} + y\dot{x}}{y} \right) :$

$Cg - GC \text{ or } Sg = \frac{anx + ny\dot{x}}{my}$. Wherefore be-

cause of the right-angled similar Triangles *FMO* and *FSg*, $MO - Sg \left(\frac{bmyx - any\dot{x} - aan\dot{x}}{amy} \right) :$

MO

$$MO \left(\frac{bx}{a} \right) : MS \text{ or } MG (b) : MF =$$

$\frac{bbmy}{bmy - any - aan}$. From whence comes out the following Construction.

Towards C make the Angle $ECH =$ FIG. 112
 GCM , and towards B take $MK = \frac{aa}{y}$. Then if you make $HK : HE :: MG : MF$. I say the Point F will be in the Caustick by Refraction.

For because of the Similarity of the Triangles $CGM, CEH, CG : CE :: n : m :: MG (b) : EH = \frac{bm}{n}$. Whence $HE - ME$ or $HM = \frac{bm - an}{n}$, $HM - MK$ or $HK = \frac{bmy - any - aan}{ny}$.

And therefore $HK \left(\frac{bmy - any - aan}{ny} \right) : HE \left(\frac{bm}{n} \right) :: MG (b) : MF = \frac{bbmy}{bmy - any - aan}$.

It is manifest, that when the Value of HK is negative, that of MF shall be so likewise: Whence it follows, that when the Point M falls between the Points G, F , the Point H falls between the Points K, E .

If the luminous Point B be next to E , or (which is the same thing) if the Curve AMD be concave next to B , then will y be changed from positive to negative; and consequently,

$$MF = \frac{-bbmy}{-bmy + any - aan} \text{ or } \frac{bbmy}{bmy - any + aan}$$

And the Construction will be the same.

If y be supposed infinite, that is, when the luminous Point B is at an infinite Distance from the Curve AMD , the incident Rays will be

M

then

then parallel, and $MF = \frac{bbm}{bm - an}$, since the Term aan is 0 with respect to bmy , any ; and because $MK \left(\frac{aa}{y} \right)$ then vanishes, we need only make $HM : HE :: MG : MF$.

COROL. I.

*Art. 114, 134. 115. **I**T may be demonstrated after the same manner, as in the Causticks by Reflection, that the Curve AMD has but one Caustick by Refraction, the Ratio of m to n being given; which Caustick is always a geometrical Curve and rectifiable, when the proposed Curve AMD is a geometrical Curve.

COROL. II.

135. **I**F the Point E falls on the other Side the Perpendicular MC with regard to the Point G , and $CE = CG$; it is evident that the Caustick by Refraction will become a Caustick by Reflection. For $MF \left(\frac{bbmy}{bmy - any + aan} \right) = \frac{ay}{2y + a}$; because $m = n$, and a is here negative and equal to b . And this agrees with what has been before demonstrated in the Section foregoing.

If m be infinite with respect to n , it is evident that the refracted or broken Ray MF will coincide with the Perpendicular CM : So that the Caustick by Refraction will be the Evolute. For MF will be $= b$, which in this Case becomes MC : That is, the Point F will coincide with the Point C , which is in the Evolute.

COROL.

COROL. III.

136. IF the convex Side of the Curve AMD be next to the luminous Point B , and the Value of $MF \left(\frac{bmy}{bmy - any - aan} \right)$ be posi-

tive; it is evident that the Point F must be assumed on the same Side of M as G is, which was supposed in the Operation of the Problem; and on the contrary, if it be negative, the same must be assumed on the contrary Side. The same is to be understood when the concave Part of the Curve AMD is towards the Point B . But we must observe that then MF

$\frac{bmy}{bmy - any - aan}$. From whence it follows, that the infinitely near refracted or broken Rays do converge when the Value of MF is positive in the first Case, and negative in the second; and on the contrary, they diverge when it is negative in the first Case, and positive in the second. This being premised, it is evident,

1^o, That if the convex Side of the Curve AMD be next to the luminous Point B , and m less than n ; or if the concave Part thereof be next to the said luminous Point, and m greater than n ; then the infinitely near refracted or broken Rays will always diverge.

2^o, If the convex Side of AMD be turned towards the luminous Point B , and m exceeds n , or the concave Side, and m be less than n ; the infinitely near refracted Rays do converge when $MK \left(\frac{aa}{y} \right)$ is less than MH

$\left(\frac{bm}{n} - a \text{ or } a - \frac{bm}{n}\right)$; diverge, when it is greater; and are parallel, when it is equal. Now since $MK=0$, when the Rays of Incidence are parallel, it follows that in this Case the infinitely near refracted Rays do always converge.

COROL. IV.

137. IF the Ray of Incidence BM touches the Curve AMD in the Point M , then will $ME(a)=0$; and therefore $MF=b$. And so the Point F will coincide with the Point G .

If the Ray of Incidence BM be perpendicular to the Curve AMD , the right Lines $ME(a)$ and $MG(b)$ will be each equal to the Radius CM of Evolution; because they coincide with it. Therefore $MF = \frac{bmy}{my - ny + bn}$, which becomes $\frac{bm}{m-n}$ when the Rays of Incidence are parallel to one another.

If the refracted Ray MF touches the Curve AMD in the Point M ; then will $MG(b)=0$. And consequently the Caustick does then touch the given Curve in the Point M .

If CM the Radius of Evolution be nothing, the straight Lines $ME(a)$, $MG(b)$ shall likewise be equal to nothing; and so the Terms aan , $bbmy$ are nothing with respect to the Terms bmy , any . Whence it follows that $MF=0$; and so the Point M is both in the Caustick and given Curve.

If CM the Radius of Evolution be infinite, the right Lines $ME(a)$, $MG(b)$ shall be infinite

finite also; and consequently the Terms bmy , any shall be nothing with regard to aan , $bbmy$:

So that MF will be $= \frac{bbmy}{+aan}$. Now since

this Quantity is * negative when the Point F * Art. 133.

falls not on the same side the Line AMD as

B ; and on the contrary is positive, when F

and B are both on the same side AMD ; there-

fore the Point F must be * assumed next to * Art. 136.

the Point B , that is, the infinitely refracted

Rays are diverging. It is evident that the

small Arch Mm then becomes a right Line,

and the Construction above will not do here.

And so we shall lay down the following one,

which determines Points in Causticks by Re-

fraction when AMD is a right Line.

Draw BO perpendicular to the Ray of In- Fig. 114.

cidence BM , intersecting the right Line MC

perpendicular to AD in the Point O ; likewise

draw OL perpendicular to the refracted Ray

MG , make the Angle BOH equal to the An-

gle LOM , and also $BM : BH :: ML : MF$.

I say, the Point F will be in the Caustick by

Refraction.

For let MC be of what Magnitude you

please, the right-angled Triangles MEC and

MBO , MGC and MLO will always be si-

milar: and therefore when it becomes infinite,

we still have $ME(a) : MG(b) :: BM(y) : ML$

$= \frac{by}{a}$. And because of the similar Triangles

OLM , OBH we have likewise $OL : OB$

$(n:m) :: ML(\frac{by}{a}) : BH = \frac{bmy}{an}$. Whence

$BM(y) : BH(\frac{bmy}{an}) :: ML(\frac{by}{a}) : MF$

$(\frac{bbmy}{aan})$.

COROLL. V.

138. **W**HEN any two of the three Points B, C, F are given, it is manifest that the third may be easily found.

EXAMPLE I.

FIG. 115. 139. **L**ET AMD be a Quadrant of a Circle, whereof the Point C is the Centre; and let the Rays of Incidence BA, BM, BD be parallel to each other, and perpendicular to CD ; and finally let the Ratio of m to n be as 3 to 2, which is the Ratio between the Sine of the Angle of Incidence and Refraction in the Passage of the Rays of Light from Air into Glass. Now because the Evolute of the Circle AMD is the Centre C thereof, if a Semicircle MEC be described, with the Radius CM , as a Diameter, and you assume the Chord $CG = \frac{2}{3}CE$, it follows that the Line MG will be the refracted Ray, in which the Point F may be determined as above (*Art. 133*.)

To find the Point H , wherein the Ray of Incidence BA perpendicular to AMD touches the Caustick by Refraction, we have * *Art. 137.* AH

$$\left(\frac{bm}{m-n} \right) = 3b = 3CA. \text{ And if a Semicircle}$$

Art. 137. CND be described with CD as a Radius, and you take the Chord $CN = \frac{2}{3}CD$; it is * manifest that the Point N will be in the Caustick by Refraction, because the Radius of Incidence BD touches the Circle AMD in the Point D .

* *Art. 132.* If AP be drawn parallel to CD , it is * manifest that the Part $FH = AH - MF - \frac{2}{3}PM$:

so that the whole Caustick $HFN = \frac{2}{3}CA -$
 $DN = \frac{7 - \sqrt{5}}{3}CA.$

If the concave Part of the Circle AMD FIG. 116.
 be turned next to the incident Rays BM , and
 the Ratio of m to n be as 2 to 3; then on the
 Semi-circumference CEM having CM as a
 Diameter, assume the Chord $CG = \frac{1}{3}CE$; and
 draw the refracted Ray MG , in which the
 Point M may be determined according to the
 general Construction (*Art.* 133.)

We shall have * $AH \left(\frac{bm}{m-n} \right) = -2b$, **Art.* 137.
 that is, AH will be * next to the convex Part **Art.* 136.
 of the Quadrant AMD , and the Double of
 the Radius AC . And if we suppose CG or
 $\frac{1}{3}CE = CM$, it is manifest that the refracted
 Ray MF will touch the Circle AMD in M ,
 because then the Point G will coincide with
 the Point M . Whence if you take $CE =$
 $\frac{2}{3}CD$, the Point M will coincide with the
 Point N , wherein the Caustick HFN touches
 * the Circle AMD : but when CE is greater **Art.* 137.
 than $\frac{2}{3}CD$, the Rays of Incidence (BM) can
 be no more refracted, that is, pass out of Glass
 into Air; because it is impossible for CG , per-
 pendicular to the refracted Ray MG , to be
 greater than CM : so that all the Rays that
 fall on the Part DN will be reflected.

If AP be drawn parallel to CD , it is ma-
 nifest * that the Part $FH = AH - MF + \frac{1}{2}$ **Art.* 137.
 PM : so that drawing NK parallel to CD ,
 the whole Caustick $HFN = 2CA + \frac{1}{2}AK =$
 $\frac{7 - \sqrt{5}}{2}CA.$

EXAMPLE II.

FIG. 117. 140. **L**ET the Curve AMD be a *Logarithmical Spiral*, the Centre thereof being the Point A , from which let all the incident Rays (AM) issue.

*Art. 91. It is manifest * that the Point E coincides with A , viz. $a=y$. Therefore if you substi-

*Art. 113. tute y for a in $\frac{bbmy}{bmy - any + aan}$ the Value * of

MF when the Concavity of the Curve is next the luminous Point; we shall have $MF=b$: Whence the Point F coincides with G .

If you draw the right Line AG , and the Tangent MT , the Angle AGO , the Complement of the Angle AGM to two right Angles, will be equal to the Angle AMT . For since the Circle, whereof CM is the Diameter, passes thro' the Points A and G , the half of the Arch AM is the Measure of each of the Angles AGO , AMT . Therefore it is evident that the Caustick AGN is a *Logarithmical Spiral*, the same as the given one, differing from it only in Position.

PROP. II.

FIG. 118. 141. **T**HE Caustick HF , the luminous Point B , and the Ratio of m to n , being given: To find any Number of Curves (as AM) whereof it may be the Caustick by Refraction.

Take the Point A at pleasure in any Tangent for one Point in the Curve AM , and from the Centre B , with the Interval BA , describe the Arch AP , and another Arch with any other Interval.

terval. Then take $AE = \frac{n}{m} PM$, and with the

Caustick HF as an Evolute, describe the Involute EM , which will intersect the Arch described with the Distance BM in the Point M ; and this will be in the Curve sought: For * $PM:AE$ or $ML::m:n$.

* Art. 132.

Otherwise.

142. IN any Tangent FM , excepting HA , find the Point M with this Condition,

that $HF + FM + \frac{n}{m} BM = HA + \frac{n}{m} BA$.

And then if you take $FK = \frac{n}{m} BA + AH -$

FH , and find the Point M such, in the Line

FK , that $MK = \frac{n}{m} BM$; the said Point will

be * that sought. Now this may be effected * Art. 132.

in describing a Curve GM of such a Nature, that the right Lines MB, MK , drawn from FIG. 119. any Point in it to the given Points B and K , may be to each other always in the constant Ratio of m to n . Now this Curve may be thus found.

Draw MR perpendicular to BK , and call the given Quantity BK, a ; and the indeterminate Quantities BR, x ; RM, y . Then because of the right-angled Triangles BRM, KRM , we shall have $BM = \sqrt{xx + yy}$, and $KM = \sqrt{aa - 2ax + xx + yy}$. So that to fulfil the Conditions of the Problem $\sqrt{xx + yy} : \sqrt{aa - 2ax + xx + yy} :: m : n$. Whence $yy = \frac{2ammx - aamm}{mm - nn} - xx$, which is a Locus ad Circulum;

culum;

culum; and may be thus described.

Assume $BG = \frac{am}{m+n}$, and $BQ = \frac{am}{m-n}$, and with GQ as a Diameter, describe the Semi-circle GMQ . I say, this will be the Locus sought. For since QR or $BQ - BR = \frac{am}{m-n} - x$, and RG or $BR - BG = x - \frac{am}{m+n}$; from the Nature of the Circle, where $QR \times RG = RM$, we have $yy = \frac{2ammx - aamm}{mm - nn} - xx$.

FIG. 120.

If the Rays of Incidence BA, BM , be parallel to a right Line given in Position, the former Solution will still take place: But the latter one is of no Effect, and instead of it we may use the following one.

Assume $FL = AH - HF$, draw LG parallel to AB , and perpendicular to AP ; assume $LO = \frac{n}{m} LG$, and draw LP parallel to GO , and PM parallel to GL . Then it is manifest *Art. 132.* that the Point M will be that sought. For since $LO = \frac{n}{m} LG$, $ML = \frac{n}{m} PM$.

If the Caustick by Refraction FH becomes a Point, the Curves (AM) will then be the famous OVALES of Descartes.

C O R O L. I.

143. AFTER the same manner as in Causticks by Reflexion, we demonstrate * that the Curves AM are of a Nature very different from one another; and that they are not geometrick Curves but when the Caustick by Refraction

fraction HF is a geometrical Curve, and rectifiable also.

COROL. II.

144. A CURVE AM , the luminous Point B , FIG. 121. and the Ratio of m to n being given: To find any Number of Lines DN of such a Nature, that they may again refract the refracted Rays MN , so as to unite them at length in a given Point C .

If we imagine the Curve HF to be the Caustick by Refraction of the given Curve AM , formed by means of the luminous Point B ; it is manifest that the said Line HF must likewise be the Caustick by Refraction of the Curve DN sought, having the given luminous Point C . Therefore $\frac{n}{m} BA + AH =$ *Art. 132.

$$\frac{n}{m} BM + MF + FH, \text{ and } NF + FH = \frac{n}{m}$$

$$NC = HD - \frac{n}{m} DC; \text{ and therefore } \frac{n}{m} BA +$$

$$AH = \frac{n}{m} BM + MN + HD - \frac{n}{m} DC +$$

$$\frac{n}{m} NC. \text{ And by the usual Transposition } \frac{n}{m}$$

$$BA - \frac{n}{m} BM + \frac{n}{m} DC + AD = MN +$$

$$\frac{n}{m} NC. \text{ Which gives the following Construction.}$$

First assume the Point D in any refracted Ray AH , as one Point in the sought Curve DN ; then on any other refracted Ray MF ,

$$\text{take the Part } MK = \frac{n}{m} BA - \frac{n}{m} BM + \frac{n}{m} DC$$

$DC + AD$; and having found the Point N

Art. 142. (as above) of such Condition, that $NK =$

*Art. 132. $\frac{n}{m} NC$, it is evident * that the same shall be in the Curve DN .

A GENERAL COROLLARY

For the three last Sections.

*Art. 80, 145. **I**T is manifest * that a Curve can have but one Evolute, one Caustick by Reflection, and one Caustick by Refraction, when the luminous Point, the Ratio of the Sines of the Angles of Incidence and Refraction is given; which Lines are always geometrical and rectifiable, when that Curve is geometrical. Whereas the same Curve may be an Evolute or a Caustick by Reflection or Refraction (in a given Ratio of the Sines, and Position of the luminous Point) to an infinite Number of Lines of different Natures, which will not be geometrical, but when that Curve is geometrical, and withal rectifiable.





S E C T. VIII.

The Use of Fluxions in finding the Points of Curves touching an infinite Number of Curves, or Right Lines given in Position.

P R O P. I. —

146. **L**ET AMB be any given Curve, whereof FIG. 122.
the right Line AP is the Axis; and let us conceive an infinite Number of Parabola's AMC , AmC , all to pass thro' the Point A , and to have the Ordinates PM , pm , as Axes. It is required to find the Curve touching all the said Parabola's.

It is manifest that the Point of Contact of every Parabola AMC , is C the Point wherein the Parabola AmC , which is infinitely near it, intersects the same. This being supposed, draw CK parallel to MP ; call the given Quantities AP, x ; PM, y ; and the unknown Quantities AK, u ; KC, z . Then from the Nature of the Parabola $\overline{AP}^2 (xx) : \overline{PK}^2 (uu - 2ux + xx)$
:: $MP (y) : MP - CK (y - z)$. Whence $zxx = 2uxy - uuy$; which is an Equation common to all the Parabola's as AMC . Here I observe that the unknown Quantities do not vary, while the given ones $AP (x)$, $PM (y)$ do, viz. become Ap and pm ; and $KC (z)$ is never

ver invariable, but when the Point C is the Intersection aforesaid: For it is manifest that every where else the right Line KC intersects the Parabola's AMC , AmC , in two different Points, and consequently it will have two different Values correspondent to the same Value of AK . Therefore if the Equation above found be thrown into Fluxions, with u and z , taken as constant Quantities, the Point C will be determined to be that of the Intersection aforesaid. Whence $2zx\dot{x} = 2ux\dot{y} + 2uy\dot{x} - u\dot{u}\dot{y}$: And so $AK(u) = \frac{2ux\dot{y} - 2y\dot{x}}{x\dot{y} - 2y\dot{x}}$, by sub-

stituting $\frac{2ux\dot{y} - u\dot{u}\dot{y}}{xx}$ for z ; and the Nature of the Curve AMB being given, we may find a Value of y in x ; which being put in the Value of AK , the said unknown Quantity will at length be expressed in known Terms freed from Fluxions. Which was proposed to be done.

If other Curves or right Lines of a determinate Position, be proposed instead of the Parabola's (AMC), the Solution of the Problem will be much the same, as will appear in the following Example.

EXAMPLE.

147. **L**ET $xx = 4ay - 4yy$ express the Nature of the Curve AMB : This will be half an Ellipsis, whose conjugate Axis is AB equal to a , and perpendicular to AP , and transverse Axis the Double of the Conjugate.

Now $xx = 2ay - 4yy$; and therefore AK
 $\left(\frac{2ux\dot{y} - 2y\dot{x}}{x\dot{y} - 2y\dot{x}} \right) = \frac{ax}{y} = u$. Whence if AK

be assumed a fourth Proportional to MP , PA , AB , and KC be drawn perpendicular to AK ; it shall intersect the Parabola AMC in the Point C sought. Again, to find the Nature of the Curve touching all the Parabola's, or which passes thro' all the Points C thus found, we must find an Equation expressing the Relation of AK (u) to KC (z) after this manner. Substi-

tute $\frac{ax}{y}$ for u its Equal in $xxx = 2uxy - uuy$,

and we get $y = \frac{aa}{2a - z}$; and therefore x or $\frac{uy}{a}$

$= \frac{au}{2a - z}$. Now if these Values be put for x

and y in $xx = 4xy - 4yy$, there will arise $uu = 4aa - 4az$, wherein x and y are got out, and which expresses the Relation of AK to KC . Therefore it is manifest, that the Curve sought is a Parabola, whereof the Line BA is the Axis, the Point B the Vertex, the Point A the Focus, and consequently the Parameter is four times AB .

We have found $y = \frac{aa}{2a - z}$, and so we get

$KC(z) = \frac{2ay - aa}{y}$. But since this Expression

is positive when $2y$ is greater than a , negative when it is less, and nothing when it is equal; therefore in the first Case the Point of Contact C falls above AP , as it was supposed to do in the Investigation; in the second, it falls below; and in the last Case it falls in AP .

If the right Line AC be drawn to intersect MP in G ; I say $MG = BQ$, and the Point G is the Focus of the Parabola AMC . For,

1^o, $AK\left(\frac{ax}{y}\right) : KC\left(\frac{2ay - aa}{y}\right) :: AP(x) :$

PG

$PG = 2y - a$. And therefore $MG = a - y = BQ$. 2°. The Parameter of the Parabola AMC , is $4a - 4y$, putting $4ay - 4yy$ for xx ; and therefore $MG = a - y$ is the fourth Part of the Parameter. Whence it is evident, that the Point G is the Focus of the Parabola; and consequently the Angle BAC must be bisected by a Tangent to the Curve in A .

Hence the Parameter of the Parabola AMC is four times BQ ; and when the Vertex M falls in A , the Parameter will be four times AB ; consequently the Parabola, whereof the Point A is the Vertex, is Asymptotick to that passing thro' all the Points C .

Because the Parabola BC touches all the Parabola's (as AMC), it is manifest that all the said Parabola's will cut the determinate Line AC in Points, which will all be nearer to A than the Point C . Now it is shewn in the Doctrine of Projecties, that (AK being a Horizontal Line) all the Parabola's, as AMC , will be the Paths of Bombs thrown out of a Mortar with a given Force placed in A , having all possible Elevations. And consequently if a right Line be drawn bisecting the Angle BAC , the same will be the Position of the Axis of the Mortar; so that a Bomb thrown out of it, will fall on the Plane AC given in Position to a Point C , which will be farther from the Mortar than when it has any other Elevation.

PROP. II.

FIG. 123. 148. **L**ET any Curve AM be given, whereof AP is its Axis: To find another Curve BC of such a Nature, that if any Ordinate PM be

be drawn, and the Perpendicular PC to the sought Curve; these Lines PM , PC , may be always equal to one another.

If an infinite Number of circular Arches be supposed to be described from the Centres P, p , with the Radii PC, pC , equal to PM, pm . It is evident that the Curve BC sought, must touch all the said Circles; and that C , the Point of Contact of every Circle, is the Point wherein the Circle infinitely near it cuts it. This being premised, draw CK perpendicular to AP , and call the given and variable Quantities AP, x ; PM or PC, y ; and the unknown and constant Quantities AK, u ; KC, z . Then from the Nature of the Circle $\overline{PC}^2 = \overline{PK}^2 + \overline{KC}^2$, viz. $yy = xx - 2ux + uu + zz$, which is the Equation common to all the said Circles. This thrown into Fluxions, will be $2y\dot{y} = 2x\dot{x} - 2u\dot{x}$; and so we get $PK(x - u) = \frac{y\dot{y}}{\dot{x}}$.

From whence comes the following general Construction.

Draw MQ perpendicular to the Curve AM , take $PK = PQ$, and draw KC parallel to PM . I say, this will meet the Circle described from the Centre P , with a Radius $PC = PM$ in the Point C , wherein it touches the Curve CB sought. This is evident, because $PQ = \frac{y\dot{y}}{\dot{x}}$.

The Value of PK may be found otherwise thus:

Draw PO perpendicular to Cp ; then the right-angled Triangles pOP , PKC , will be
Nsimilar.

similar. Therefore $Pp(\dot{x}):Op(\dot{y})::PC(y):$
 $PK=\frac{yy}{\dot{x}}$.

When $PQ=PM$, it is manifest that the Circle described with the Radius PC , will touch KC in the Point K ; so that the Point C will then coincide with K , and consequently will fall in the Axis.

But when PQ is greater than PM , the Circle described with the Radius PC , cannot touch the Curve BC ; because it cannot at all intersect the right Line KC .

EXAMPLE.

FIG. 123. 149. **L**ET the given Curve AM be a Parabola, the Equation whereof is $ax=yy$. Now PQ or $PK(x-u)=\frac{1}{2}a$, and consequently $x=\frac{1}{2}a+u$, and $yy=\frac{1}{4}aa+zz$, because of the right-angled Triangle PKC . If these Values be put in $ax=yy$, we shall have $\frac{1}{2}aa+au=\frac{1}{4}aa+zz$, or $\frac{1}{4}aa+au=zz$, expressing the Nature of the Curve BC . Whence we may perceive that the said Curve is also a common Parabola as well as AM , because they have both the same Parameter a , and the Vertex B is distant from the Vertex A , by the Distance $BA=\frac{1}{4}a$.

PROP. III.

FIG. 124. 150. **L**ET AM be any given Curve, the right Line AP being a Diameter, and the Ordinates PM, pm , parallel to a right Line AQ given in Position; and having drawn MQ, mq , parallel to AP , then draw the right Lines PQC, pqC . It is required to find the Curve AC

AC of such a Nature, that all these last mentioned Lines may be Tangents too; or, which is the same thing, to find the Point of Contact C in every right Line PQC.

Conceive another Tangent pqC to be infinitely near PQC , and draw CK parallel to AQ ; then call the given and variable Quantities AP, x ; PM or AQ, y ; the unknown and variable Quantities AK, u ; KC, z . Now because of the similar Triangles PAQ, PKC , $AP(x) : AQ(y) :: PK(x+u) : KC(z) = y + \frac{uy}{x}$. Which is an Equation common to all the right Lines as KC . The Fluxion thereof is $\dot{y} + \frac{ux\dot{y} - uy\dot{x}}{xx} = 0$. From whence arises AK

$(u) = \frac{xx\dot{y}}{y\dot{x} - x\dot{y}}$; and so the following general Construction.

Draw the Tangent MT , in which assume AK a third Proportional to AT, AP : Then if KC be drawn parallel to AQ , it shall cut the right Line PQC in the Point C sought.

$$\text{For } AT\left(\frac{y\dot{x} - x\dot{y}}{\dot{y}}\right) : AP(x) :: AP(x) : AK \\ = \frac{xx\dot{y}}{y\dot{x} - x\dot{y}}.$$

EXAMPLE I.

151. **L**ET the given Curve AM be a Para- FIG. 124.
bola; so that the Equation thereof will be $ax = yy$. Then $AT = AP$; whence $AK(u) = x$; that is, the Point K falls in T . Now to have an Equation expressing the Relation of $AK(u)$ to $KC(z)$; we shall have $KC(z) = 2y$, because PK is the Double of AP .

N 2

Now

Now putting u and $\frac{1}{2}z$ for their Equals x and y in $ax=yy$, and then will $4au=zz$: Whence we may perceive that the Curve AC is a Parabola, whose Vertex is A , and Parameter a Line equal to four times the Parameter of the Parabola AM .

EXAMPLE II.

FIG. 125. 152. **L**ET the given Curve AM be a Quadrant BMD , whose Centre is the Point A , and Semidiameter the Line AB or AD , which I call a . It is manifest that PQ is always equal to AM or AB , viz. is an invariable Quantity. So that the Ends P, Q may be supposed to slide or move along the Sides BA, AD of the right Angle BAD . Now $AK(u)$ will be $=\frac{x^3}{aa}$, because $AT=\frac{aa}{x}$; and on account of the Parallels KC, AQ ; $AP(x):PQ(a)::AK\left(\frac{x^3}{aa}\right):QC=\frac{xx}{a}$. Whence we may perceive that to determine the Point of Contact C ; we need only assume QC a third Proportional to PQ and AP . If you seek the Equation of the Curve BCD , the same will be found this here, $u^6-3aa u^4+3a^4uu-a^6=0$.

$$\begin{array}{r}
 +3zz \\
 +21aazz+3a^4zz \\
 +3z^4-3aaz^4 \\
 +z^6
 \end{array}$$

COROL. I.

153. **I**F the Relation of DC the Part of the Curve BCD to its Tangent CP be required, you must first conceive another Tangent cp infinitely near CP ; then describing the

the little Arch PO from the Centre C , and $cp - CP$ or $Op - Cc$ will be $= \frac{2x\dot{x}}{a}$, which

is the Fluxion of $CP = \frac{aa - xx}{a}$: whence

there comes out $Cc = Op + \frac{2x\dot{x}}{a}$. Now be-

cause of the right-angled similar Triangles $\triangle PQA, PpO, P\mathcal{Q}(a) : AP(x) :: Pp(x) : Op = \frac{x\dot{x}}{a}$. And therefore $Cc = \frac{3x\dot{x}}{a} = DC -$

Dc . Whence it is evident, that where-ever the Point C be assumed, this Proportion will always be had, viz. $DC - Dc \left(\frac{3x\dot{x}}{a} \right) : CP$

$- cp \left(\frac{2x\dot{x}}{a} \right) :: 3 : 2$. And consequently the

Sum of all the Fluxions $DC - Dc$ corresponding to the same right Line PD , that is, * the * *Art. 96.* Part DC of the Curve (BCD) is to the Sum of all the Fluxions $CP - cp$ answerable to the same right Line PD , viz. * to the Tangent * *Art. 96.* $CP :: 3 : 2$. And also the whole Curve BCD is to its Tangent $BA :: 3 : 2$.

COROL. II.

154. IF the Curve BCD be taken as an Evolute, the Involute DNF formed from it beginning at D , will be of such a Nature, that $CN : CP :: 3 : 2$. Because CN is always equal to DC the Part of the Curve BCD . Whence it follows, that the similar Sectors CNn, CPO , are to one another $:: 9 : 4$. And therefore the Space DCN comprehended under the Curves DC, DN , and the right Line CN , which is

N 3

a Tan-

a Tangent in C , and perpendicular at N , is to the Space DCP , contained under the Curve DC and the Tangents DP, CP , as 9 to 4.

C O R O L. III.

155. **T**HE Centre of Gravity of the Sector CNn must be in the Arch PO ; because $CP = \frac{2}{3}CN$. And since the said Arch is infinitely small, it follows that the Centre of Gravity must be in the right Line AD ; and therefore the Centre of Gravity of the Spaces DCN, BDF , made up of all those Sectors, must be in the right Line AD . Consequently if a Figure be described on the other side of BDF equal and similar to BDF , the Centre of Gravity of the entire Figure shall be in the Point A .

C O R O L. IV.

156. **B**ECAUSE of the right-angled similar Triangles $PQA, pPO, PQ(a) : AQ$ or $PM (\sqrt{aa-xx}) :: Pp(x) : PO = \frac{x}{\sqrt{aa-xx}}$. And because of the similar Sectors

$CPO, CNn, CP : CN$ or $2 : 3 :: PO (\frac{x}{\sqrt{aa-xx}}) : Nn = \frac{3}{2} \frac{x}{\sqrt{aa-xx}}$. Now the

* Art. 2. Rectangle $MP \times Pp$, viz. * the little circular Space $MPpm = \frac{x}{2} \sqrt{aa-xx}$. Whence $AB \times Nn = \frac{3}{2} MPpm$: and consequently the Part ND of the Curve DNF drawn into the Radius AB , is the Sesquialter of the circular Segment DMP , and the whole Curve DNF is equal to $\frac{3}{2}$ of the circular Quadrant BMD .

PROP.

PROP. IV.

157. **L**ET *AM* be any given Curve, whereof FIG. 126.
the right Line *AP* is the Axis; and
let there be an infinite Number of Perpendiculars
MC m C drawn to the same. It is required to
find that Curve which all these Perpendiculars
are Tangents to; or, which is the same thing, to
find the Point of Contact *C* in every Perpendicu-
lar *MC*.

First imagine another Perpendicular *m C* in-
finitely near to *MC*; let *MP* be an Ordinate,
and through the Point of Intersection *C* draw
the right Lines *CK* perpendicular, and *CE*
parallel to the Axis: then call the given and
variable Quantities *AP, x*; *PM, y*; and the
unknown and invariable ones *AK, u*; *KC, z*.

This done, *PQ* will be $= \frac{y\dot{y}}{\dot{x}}$, *PK* or *CE* =

u - x, *ME* = *y + z*; and because of the right-
angled similar Triangles *MPQ*, *MEC*, *MP*

$$(y) : P Q \left(\frac{y\dot{y}}{\dot{x}} \right) :: ME (y + z) : EC (u - x)$$

$$= \frac{y\dot{y} + z\dot{y}}{\dot{x}}. \text{ Which is an Equation common}$$

to all the Perpendiculars as *MC*, and the Flu-
xion thereof (supposing $\dot{x}$ invariable) will be

$$-\dot{x} = \frac{y\ddot{y} + \dot{y}^2 + z\ddot{y}}{\dot{x}}; \text{ from whence comes out}$$

$$ME (z + y) = \frac{\dot{x}^2 + \dot{y}^2}{-\dot{y}}. \text{ Now the Nature of}$$

the Curve *AM* being given, we shall have
Values of $\dot{y}^2$ and $\ddot{y}$ in $\dot{x}^2$, which being put in

$$\frac{\dot{x}^2 + \dot{y}^2}{-\dot{y}}, \text{ will give a known Value of } ME$$

freed from Fluxions; which is what was proposed.

It is manifest that the Curve passing thro' all the Points C , is the Evolute of the Curve AM : and because in the fifth Section these Curves are fully handled, it is needless to give again new Examples of them here.

P R O P. V.

FIG. 127. 158. *ANY two Lines AM , BN being given, together with a right Line MN continuing always of the same Length. Now if the Ends M , N of this Line continually move along the two former Lines, it is required to find the Curve that it always touches.*

First draw the Tangents MT , NT , and conceive another right Line mn infinitely near MN , and which by consequence cuts it in the Point C , wherein the same touches the Curve in which we are now determining Points. Now it is plain, that while the right Line MN is moving to the Situation mn , the Ends thereof will describe, or continually be in the small Parts Mm , Nn of AM , BN , which on account of their being infinitely small, are common to the Tangents TM , TN : So that while the right Line MN is moving to the infinitely-near Situation mn , the Ends thereof may be conceived as moving along the right Lines TM , TN given in Position.

This being well understood, draw MP , CK perpendicular to NT , and call the given and variable Quantities TP , x ; PM , y ; the unknown and constant Quantities TK , u ; KC , z ; and the stable Line MN , a . Now because
of

of the right-angled Triangle MPN , $PN = \sqrt{aa - yy}$; and because of the similar Triangles NPM , NKC , $NP(\sqrt{aa - yy}) : PM(y) :: NK(u - x - \sqrt{aa - yy}) : KC(z) = \frac{uy - xy}{\sqrt{aa - yy}} - y$. And this thrown into Fluxions will be $aa\dot{u}\dot{y} - aa\dot{x}\dot{y} - aa\dot{y}\dot{x} + y^3\dot{x} = aa\dot{y} - yy\dot{y}\sqrt{aa - xx}$: and making $\sqrt{aa - yy} = m$ for brevity's sake, there comes out $PK(u - x) = \frac{m^3\dot{y} + mmy\dot{x}}{aa\dot{y}} = \frac{m^3 + mmx}{aa}$, by substituting $x\dot{y}$ for its Equal $y\dot{x}$, because of the similar Triangles mRM , MPT ; and therefore $MC = \frac{mm + mx}{a}$: from whence comes the following Construction.

Draw TE perpendicular to MN , and assume $MC = NE$: I say, the Point C will be that sought. For since the right-angled Triangles MNP , TNE are similar, $MN(a) : NP(m) :: NT(m + x) : NE$ or $MC = \frac{mm + mx}{a}$.

Otherwise. Draw TE perpendicular to MN , and describe the small Arches MS , NO , from the Centre C , call the given Quantities NE, r ; ET, s ; MN, a ; and the unknown Quantity CM, t . Then will Sm or $On = i$; and because of the right-angled similar Triangles MET and mSM , NET and nON , CMS and CNO , $ME(r - a) : ET(s) :: mS(i) : SM = \frac{si}{r - a}$. And $NE(r) : ET(s) :: nO(i) : ON = \frac{si}{r}$. And $MS - NO \left(\frac{ast}{rr - ra} \right) : MS \left(\frac{si}{r - a} \right) :: MN(a) : MC(t) = r$. From whence arises the same Construction as above. If

If AM, BN be supposed right Lines at right Angles to each other; it is manifest that the Curve sought is the same as that of Article 152.

PROP. VI.

FIG. 128. 159. **L**ET L, M, N be any three given Lines, and from every Point L, l in the Curve L , let two Tangents LM and LN , lm and ln be conceived to issue, to the Curves M and N , one to each. It is required to find a fourth Curve C , to which all the right Lines MN, mn joining the Points of Contact of the Curves M, N may be Tangents.

Draw the Tangent LE , and thro' any Point E therein draw EF, EG perpendicular to the two other Tangents ML, NL , let the Point l be supposed infinitely near L , draw the little right Lines LH, LK perpendicular to ml, nl ; as likewise MP, mP, NQ, nQ perpendicular to the Tangents ML, ml, NL, nl , which will intersect each other in the Points P and Q . Then the right-angled Triangles EFL and LHl, EGL and LKl will be similar; as likewise the Triangles LMH and MPm, LnK and NQn right-angled at H and m, K and N ; because each of the Angles LMH, MPm added to the Angle PMm , makes a right Angle. And in like manner we prove that the Angles LnK, NQn are equal to one another.

This being premised, call the little side Mm of the Polygon, the Curve M is conceived to be, u ; and the given Quantities EF, m ; EG, n ; MN or mn, a ; ML or ml, b ; NL

NL or nl, c ; MP or mP, f ; NQ or nQ, g ;
(I here take the right Lines MP , NQ for
given ones, because the Nature of the Curves
 M, N being given by Supposition, they may
always *be found); then we shall have, 1°, MP * *Art. 78.*

$$(f):ML(b)::Mm(\dot{u}):LH=\frac{b\dot{u}}{f}. \quad 2^\circ, EF(m)$$

$$:EG(n)::LH\left(\frac{b\dot{u}}{f}\right):LK=\frac{bn\dot{u}}{mf}. \quad 3^\circ, LN$$

$$\text{or } Ln(c):nQ(g)::LK\left(\frac{bn\dot{u}}{mf}\right):nN=\frac{bg n\dot{u}}{cfm}.$$

4°, (Drawing MR parallel to NL or nl) ml

$$(b):ln(c)::mM(\dot{u}):MR=\frac{c\dot{u}}{b}. \quad 5^\circ, MR+$$

$$Nn\left(\frac{c\dot{u}}{b}+\frac{bg n\dot{u}}{cfm}\right):MR\left(\frac{c\dot{u}}{b}\right)::MN(a):$$

$$MC=\frac{accfm}{ccfm+bbgn}, \text{ which was to be found.}$$

When the Tangent EL coincides with the
Tangent ML , it is manifest that $EF(m)$ will
become nothing; and therefore the Point
sought C will coincide with M . Likewise
when the Tangent EL coincides with the
Tangent LN ; then $EG(n)$ will become no-
thing; and consequently $MC=a$. Whence
it is evident, that the Point C sought will co-
incide also with N . Lastly, if the Tangent
 EL falls in the Angle GLI ; in this Case EG
(n) will be negative: so that then $MC=$

$$\frac{accfm}{ccfm-bbgn}; \text{ and the Point sought } C \text{ will fall}$$

without the Points M and N .

EXAM-

EXAMPLE I.

FIG. 129. 160. **S**UPPOSE the Curves M, N , to be Parts of the same Circle; then it is plain, that in this Case $b=c$, and $f=g$: So that $MC = \frac{am}{m+n}$; whence it follows, that the Point C sought, is determined by dividing the right Line MN into two Parts, being to one another in the given Ratio of m to n , viz. so that $MC : NC :: m : n$:

EXAMPLE II.

161. **S**UPPOSE the Curves M and N be any Conick Section. Here the general Construction will be changed into a far more simple one, from the Consideration of a Property of the Conick Sections demonstrated in Treatises of those Curves, viz. That if from every Point L, l , of a right Line EL , be drawn two Tangents LM and LN , lm, ln , to any Conick Section, all the Right Lines MN, mn , joining the Points of Contact, will intersect one another in one and the same Point C , thro' which the Diameter AC , to Ordinates that are parallel to the Right Line EL passes. For it follows from thence, that the Point C is determined by only drawing a Diameter whose Ordinates are parallel to the Tangent EL .

In the Circle it is manifest that the Diameter must be perpendicular to the Tangent EL ; that is, a Perpendicular AB drawn from the Centre A to the Tangent, will intersect the right Line MN in the Point C sought.

S C H O

SCHOLIUM.

162. **T**HE Solution of the following Problem depending on the Method of Tangents, will be had by means of the afore-said Problem. FIG. 128.

The three Curves C, M, N , being given, and a right Line MN being continually moved about the Curve C , so as to always touch it: And if from the Points M, N , wherein it cuts the Curves M and N , be drawn the Tangents ML, NL , intersecting each other in the Point L , which by the Motion describes a fourth Curve Ll . It is required to draw LE the Tangent to this Curve, the right Lines MN, ML , together with the Point of Contact C being given.

For it is manifest that this Problem is but the Inverse of the foregoing one, and here MC is given: So that we are to find the Ratio of EF to EG , which determines the Position of the Tangent EL . Therefore if you call the given Quantity MC, b , we shall have

$$\frac{accfm}{ccfm + bbgn} = b. \text{ Whence comes out } m =$$

$$\frac{bbghn}{accf - ccfb}; \text{ and consequently the Tangent } LE$$

must be so situate in the given Angle MLG , that if from any Point E in it, be drawn EF, EG , perpendicular to the Sides of that Angle, they may be always to each other in a given Ratio, viz. of $bbgh$ to $accf - ccfb$. Now this is done by drawing MD parallel to NL and

$$= \left(\frac{bbgh}{accf - ccfb} \right).$$

When

FIG. 129.

*Art. 161.

When the Curves M and N , are Parts of the same Conick Section, it is * manifest that then you need but draw the Tangent LE parallel to the Ordinates to the Diameter passing thro' the Point C .



S E C T.



S E C T. IX.

The Solution of some Problems depending on the Methods aforegoing.

P R O P. I.

163. **L**ET AMD be a Curve ($AP = x$, $PM = y$, $AB = a$) of such a Nature, that the Value of the Ordinate y is expressed by a Fraction, the Numerator and Denominator of which, do each of them become 0 when $x = a$, viz. when the Point P coincides with the given Point B . It is required to find what will then be the Value of the Ordinate BD . FIG. 130.

Let ANB , COB , be two Curves (having the Line AB as a common Axis) of such a Nature, that the Ordinate PN expresses the Numerator, and the Ordinate PO the Denominator of the general Fraction representing any Ordinate PM : So that $PM = \frac{AB \times PN}{PO}$.

Then it is manifest, that these two Curves will meet one another in the Point B ; since by the Supposition PN , PO , do each become 0 when the Point P falls in B . This being supposed, if an Ordinate bd be imagined infinitely near to BD , cutting the Curves ANB , COB ,

** Art. 2.* COB , in the Points f, g ; then will $bd = \frac{AB \times bf}{bg}$, which will be ^{*}equal to BD . Now our Business is only to find the Relation of bg to bf . In order thereto it is manifest, when the Absciss AP becomes AB , the Ordinates PN, PO , will be 0, and when AP becomes Ab , they do become bf, bg . Whence it follows, that the said Ordinates bf, bg , themselves, are the Fluxions of the Ordinates in B and b , with regard to the Curves ANB, COB ; and consequently, if the Fluxion of the Numerator be found, and that be divided by the Fluxion of the Denominator, after having made $x = a = Ab$ or AB , we shall have the Value of the Ordinates bd or BD sought. Which was to be found.

EXAMPLE I.

164. **L**_{ET} $y = \frac{\sqrt{2a^3x - x^4} - a^{\frac{3}{2}}\sqrt{ax}}{a - \sqrt[4]{ax^3}}$. Now

it is manifest when $x = a$, that the Numerator and Denominator of the Fraction will each be equal to 0. Therefore we must assume the Fluxion of the Numerator $\frac{a^3\dot{x} - 2x^3\dot{x}}{\sqrt{2a^3x - x^4}}$

$-\frac{aa\dot{x}}{3\sqrt[3]{aax}}$, and divide it by the Fluxion of the

Denominator $-\frac{3a\dot{x}}{4\sqrt[4]{a^3x}}$, after having made $x = a$, viz. divide $-\frac{1}{3}a\dot{x}$ by $-\frac{1}{4}\dot{x}$; and there comes out $\frac{1}{3}a = BD$.

E X A M-

EXAMPLE II.

165. **L**ET $y = \frac{aa - ax}{a - \sqrt{ax}}$. Then when $x = a$,
 y will be $= 2a$.

This Example may be solved without Fluxions thus.

Taking away the Surds, and then will $aaax + 2aaxy - axyy - 2a^3x + a^4 + aayy - 2a^3y = 0$, which being divided by $x - a$, will be brought down to $aax - x^3 + 2aay - ayy = 0$; and substituting a for x , we have as before $y = 2a$.

LEMMA.

166. **L**ET BCG be any Curve, which the right FIG. 131.

Line AE touches in the Point B , in which take two Points (having an invariable Position) A, E . Now if this right Line moves about the Curve so as to touch it continually, it is evident that the stable Points A, E , by the said Motion will describe two Curves AMD, ENH . Then if DL be drawn parallel to AB , and which consequently makes with DK (in which the right Line AE is supposed to be when it touches the Curve BCG in G) the Angle KDL equal to the Angle AOD formed by the Tangents in B and D ; and from the Centre D be described any Arch KFL .

Ifay $DK : KFL :: AE : AMD \pm ENH$, viz. $+$ when the Point of Contact falls between the describent Points, and $-$ when it does not.

For suppose the right Line AE in its Motion about the Curve BCG to be come to the Situations or Positions MCN, mCn infinitely
○
near

near each other, and draw the Radii DF, Df , parallel to CM, Cm . Then it is manifest that the Sectors DFf, CMm, CNn , are similar; and so $DF : Ff :: CM : Mm :: CN : Nn :: CM + CN$ or $AE : Mm + Nn$. Now since this is always so, let the Point of Contact C be any where taken, it follows that the Radius DK is to the Arch KFL the Sum of all the small Arches $Ff : AE :: AMD + ENH$ the Sum of all the little Arches $Mm + Nn$. Which was to be demonstrated.

COROL. I.

167. **I**T is plain that the Curves AMD, ENH , are formed by the Evolution of the Curve BCG ; and so the right Line AE is always perpendicular to those two Curves in all the various Positions of it; so that their Distance is every where the same; which is the Nature of parallel Lines. Whence it appears that a Curve Line AMD being given, we can find an infinite Number of Points in the Curve ENH without the use of the Evolute BCG , by drawing an infinite Number of Perpendiculars to that Curve, and taking them all equal to the right Line AE .

COROL. II.

168. **I**F BC, CG , the Halves of the Curve BCG , be similar and equal, it is manifest that the Curves AMD, ENH , shall be similar and equal; so that they only differ in Position. Whence it follows, that the Curve AMD will be to the circular Arch $KFL :: AE : DK$; That is, in a given Ratio.

PROP.

PROP. II.

169. **L**ET there be any two Curves AEV , BCG , together with a third Curve AMD of such a Nature, that a Part of a Curve EM being described from the Evolution of the Curve BCG , the Relation of the Portions or Parts AE , EM , to the Radii of Evolution EC , MG , be expressed by any given Equation. It is required to draw the Tangent MT from the given Point M in the Curve AMD . FIG. 132.

Conceive another Part or Portion em of the Curve infinitely near to EM , and the Radii of Evolution CeF , GmR , to be drawn. Then,
 1°, Let CH be perpendicular to CE , meeting EH the Tangent to the Curve AEV in H .
 2°, ML parallel to CE , meeting the Arch GL described from the Centre M with the Radius MG in the Point L .
 3°, GC be perpendicular to MG meeting the sought Tangent MT in T .

This being done, make $AE=x$, $EM=y$, $CE=u$, $GM=z$, $CH=s$, $EH=t$, the Arch $GL=r$. Then will $Ee=\dot{x}$, Fe or $Rm=\dot{u}$ $=\dot{z}$; and because of the right-angled similar Triangles eFE and ECH , $CE(u):CH(s)::$

$$Fe(\dot{z}):FE=\frac{s\dot{z}}{u}. \text{ And } CE(u):EH(t)::$$

$$Fe(\dot{z}):Ee(\dot{x})=\frac{t\dot{z}}{u}. \text{ Now by the Lemma*}$$

$$RF-me=\frac{r\dot{z}}{z}; \text{ and therefore } RM(\overline{RF-me}$$

$$+\overline{me-ME}+\overline{ME-MF})=\frac{r\dot{z}}{z}+\dot{y}+\frac{s\dot{z}}{u}.$$

Whence because of the similar Triangles mRM ,

$$MGT, mR(z) : RM\left(\frac{r\dot{z}}{z} + \frac{s\dot{z}}{u} + \dot{y}\right) ::$$

$$MG(z) : GT = r + \frac{s\dot{z}}{u} + \frac{z\dot{y}}{\dot{z}}; \text{ but if } \dot{z} \text{ and } \frac{t\dot{z}}{u}$$

be put for their Equals u and $\dot{z}$ in the Fluxion of the given Equation, we shall get a Value of $\dot{y}$ in $\dot{z}$; which being substituted in $\frac{z\dot{y}}{\dot{z}}$, there will come out a known Value (freed from Fluxions) of the Subtangent GT sought. Which was to be done.

FIG. 133. If the Curve BCG degenerates into a Point O ; it is manifest that the Portion of the Curve $ME(y)$ will then be an Arch of a Circle equal to the Arch $GL(r)$, and the Radii $CE(u)$, $GM(z)$ of Evolution will be equal to each other: So that GT , which in this Case becomes OT , will be $= y + s + \frac{z\dot{y}}{\dot{z}}$.

EXAMPLE.

FIG. 133 170. LET $y = \frac{xz}{a}$; this thrown into Fluxions

will be $\dot{y} = \frac{z\dot{x} - x\dot{z}}{a}$ ($-x\dot{z}$ being taken negative

* Art. 8. *, because while x and y increase, z decreases) $= \frac{t\dot{z} - x\dot{z}}{a}$, by substituting $\frac{t\dot{z}}{z}$ for its Equal $\dot{x}$;

and therefore $OT\left(y + s + \frac{z\dot{y}}{\dot{z}}\right) = y + s +$

$\frac{t\dot{z} - x\dot{z}}{a} = \frac{as + t\dot{z}}{a}$, putting y for its Equal $\frac{xz}{a}$.

SCHO-

SCHOLIUM.

171. IF the Point O falls in the Axis AB , and FIG. 134.
the Curve AEV be a Semicircle, the
Curve AMD will be half a Cycloid genera-
ted by the Rotation of a Semicircle BSN a-
long an equal Arch BGN of a Circle descri-
bed from the Centre O , the generating Point
 A falling without, within, or upon the Cir-
cumference of the moveable Circle BSN , ac-
cording as the given Quantity a is greater,
equal to, or less than OV . To prove this, and
withal determine the Point B ,

Let us suppose the thing to be so, viz. that
the Curve AMD is a Semi-cycloid, genera-
ted by the Rotation of the Semicircle BSN ,
(whose Centre is K the Centre of the Semi-
circle AEV) along the Arch BGN described
from the Centre O ; and conceiving the said
Semicircle BSN to remain in such a Situation
 BGN , that the describing Point A falls in the
Point M , draw the right Line OK thro' the
Centres of the generating Circles; which, by
consequence, will pass thro' the Point of Con-
tact G ; then drawing KSE , we may observe
that the Triangles OKE , OKM , are equal
and similar, because the three Sides of the one
are each equal to the three Sides of the other.
Whence it follows, 1^o, That the extreme An-
gles MOK , EOK , are equal; and so likewise
the Angles MOE , GOB . Whence GB :
 $ME :: OB:OE$. 2^o, That the Angles MKO ,
 EKO , are moreover equal: And consequently
the Arches GN , BS , measuring them, are e-
qual also. The same may be said of their Com-
plements GB , SN , to two right Angles, be-

cause they appertain to equal Circles. Now by the Generation of the Cycloid, the Arch GB of the moveable Circle is equal to the Arch GB of the immoveable one. Wherefore $SN:ME::OB:OE$. This being premised,

Call the known Lines OV, b ; KV or KA, c ; and the unknown one KB, u . Then will $OB = b + c - u$; and because of the similar Sectors $KEA, KSN, KE(c):KS(u)::AE(x):SN = \frac{ux}{c}$. And therefore $OB(b + c - u)$

$$:OE(z)::SN\left(\frac{ux}{c}\right):EM(y) = \frac{uxz}{bc + cc - cu} \\ = \frac{xz}{a}. \text{ Whence we get } KB(u) = \frac{bc + cc}{a + c}.$$

Wherefore if you assume $KB = \frac{bc + cc}{a + c}$, and from the Centres K and O do describe the Semicircle BSN and Arch BGN ; it is evident that the Curve AMD will be half a Cycloid, described by the Rotation of the Semicircle BSN along the Arch BGN , the describing Point A falling without, within, or on the Circumference of the said Circle, according as $KV(c)$ is greater, less than, or equal to $KB\left(\frac{bc + cc}{a + c}\right)$, that is, according as a is greater, less than, or equal to $OV(b)$.

COROL. I.

172. **I**t is manifest that $EM(y):AE(x)::KB \times OE(uz):OB \times KV(bc + cc - uc)$. Now if OB be supposed to become infinite, the right Line OE will be so also, and parallel to OB , because it will never meet the same, the

the Concentrick Arches BGN , EM will become parallel right Lines, perpendicular to OB , OE ; and the right Line EM will be to the Arch $AE::KB:KV$. Because the infinite right Lines OE , OB , which differ from each other by a finite Magnitude only, may be looked upon as equal.

COROL. II.

173. **B**ECAUSE the Angles MKO , EKO are equal, the Triangles MKG , EKB shall be equal and similar; and so the right Lines MG , EB are equal to each other. Whence * if it be required to draw MG from * Art. 43. a given Point M in the Curve of the Cycloid perpendicular to the same, you need only describe the Arch ME from the Centre O , and from the Centre M with the Distance EB , an Arch which will cut the Base BGN in the Point G , thro' which and the given Point M you may draw the Perpendicular requir'd.

COROL. III.

174. **T**HE Point G being given in the Circumference of the moveable Semi-circle BGN ; and it be requir'd to find the Point M in the Cycloid wherein the describing Point A falls, when the given Point G touches the Base; you must assume the Arch $SN=BG$, and drawing the Radius KS meeting the Circumference AEV in E , describe the Arch EM from the Centre O . Then it is manifest that the said Arch shall cut the Cycloid in the Point M sought.

P R O P. III.

FIG. 135, 175. **L**ET AMD be a Semi-cycloid described
 136. by the Rotation of the Semi-circle BGN
 along an Arch BGN equal to it; so that the
 contiguous Parts BG, BG, as they still increase,
 are constantly equal to one another: And let the
 describing Point M be assumed in the Diameter
 BN, either without, within, or on the Circum-
 ference of the moveable Circle BGN. It is re-
 quir'd to find the Point M in the Semi-cycloid,
 whose Distance from the Axis OA thereof shall
 be a maximum.

Art. 47. If the Point M be supposed to be that sought,
 it is * manifest that the Tangent in M will be
 parallel to the Axis OA; and therefore the
 Perpendicular MG to the Cycloid, must like-
 wise be perpendicular to the Axis which it
 meets in the Point P. This being supposed,
 if OK be drawn thro' the Centres of the ge-
 neraling Circles, it will pass thro' the Point
 of Contact G; and if KL be drawn perpen-
 dicular to MG, the equal Angles GKL, GOB
 will be formed; and therefore the Arch IG,
 which is the Double of the Measure of the
 Angle GKL, will be to the Arch GB, the
 Measure of the Angle GOB, as the Diameter
 BN is to the Radius OB. Whence the De-
 termination of the Point G, in the Arch of
 the Semicircle BGN, wherein it touches the
 Arch serving as a Base to it, when the Distance
 of the describing Point M from AO is a
maximum, will be had by so dividing the Semi-
 circle BGN in G, that drawing the Chord
 IG thro' the given Point M, the Arch IG is
 to

to the Arch BG , in the given Ratio of BN to OB . Therefore the Problem is brought to a common geometrical one, and may be always solved geometrically, when the given Ratio can be expressed in whole Numbers; but by means of Lines represented by Equations of so many Dimensions the more, as the Ratio is more compounded.

If the Radius OB be supposed to become infinite, as it will be when the Base BGN is a right Line; then the Arch IG shall be infinitely small with respect to the Arch GB . Therefore the Secant MIG then will become the Tangent MT , when the describing Point M falls without the moveable Circle; and it is manifest that when the describing Point M falls within the Circle, there will be no *maximum* in the Case aforesaid.

When the Point M falls in N on the Circumference, you need only divide the Semicircumference BGN in the Point G in the given Ratio of BN to OB . For the Point G so found will be that wherein the moveable Circle BGN touches the Base, when the describing Point falls in the Point sought.

LEMMA.

176. **I**N every Triangle BAC , the Angles FIG. 137.
thereof ABC , ACB , and CAD the
Complement of the obtuse Angle BAC to two
right Angles being infinitely small, are in the same
Proportion as the opposite Sides AC , AB , BC .

For if a Circle be described about the Triangle BAC , the Arches AC , AB , BAC being the Measures of the Doubles of those Angles,

gles, will be infinitely small; and so will not
 * Art. 3. * differ from their Chords or Subtenses.

If the Sides AC , AB , BC of the Triangle BAC be not infinitely little, viz. are of a finite Magnitude; it follows, that the Circumscribing Circle must be infinitely great; because the Arches AC , AB , BAC having a finite Magnitude, and being the Measures of the infinitely small Angles, must be infinitely small with respect to that Circle.

PROP. IV.

FIG. 135. 177. **T**HE same Things being supposed; it is
 136. required to determine the Point C in every Perpendicular MG , wherein it touches the Evolute of the Cycloid.

First conceive another Perpendicular mg infinitely near MG , and which by consequence cuts the same in the sought Point C , draw the right Line Gm , assume the small Arch Gg in the Circumference of the moveable Circle, equal to the Arch Gg in the moveable Circle, and draw the right Lines Mg , Ig , Kg , Og . This being done, if the small Arches Gg , Gg be conceived as straight Lines perpendicular to the Radii Kg , Og , it is evident that when the little Arch Gg in the moveable Circle coincides with the Arch Gg of the immoveable one, the describing Point M will coincide with m ; so that the Triangle GMg will coincide with the Triangle Gmg . Whence it appears that the Angle MGM is equal to the Angle $gGg = GKg + GOg$; because the same Angles KGg , OGg being added to both Sides, will make up two right Angles.

Now

Now calling the given Lines OG, b ; KG, a ; GM or Gm, m ; GI or Ig, n ; and we shall have, * 1°, $OG : KG :: GKg : GOg$. And * *Art. 176.* $OG (b) : OG + GK$ or $OK (b + a) :: GKg :$

$$GKg + GOg \text{ or } MGm = \frac{a+b}{b} \times GKg.$$

2°, * $Ig : MI :: GMg : MgI$. And $Ig \pm MI$ * *Ibid.* or $MG(m) : Ig(n) :: GMg \pm MgI$ or $G \bar{I}g$, or

$$\frac{1}{2} GKg : GMg \text{ or } Gmg = \frac{n}{2m} GKg. \quad 3^\circ, \text{ The}$$

Angle * MCm or $MGm - Gmg \left(\frac{a+b}{b} - \frac{n}{2m} \right)$ * *Ibid.*

$$GKg : Gmg \left(\frac{n}{2m} GKg \right) :: Gm(m) : GC =$$

$\frac{bm n}{2am + 2bm - bn}$. And consequently the Radius MC of Evolution sought will be =

$$\frac{2amm + 2bmm}{2am + 2bm - bn}.$$

If the Radius $OG (b)$ of the immoveable Circle be supposed to become infinite, the Circumference thereof will be a right Line; and striking out the Terms $2amm, 2am$, as being infinitely little with respect to the others $2bmm, 2bm - bn$, we shall have $MC =$

$$\frac{2mm}{2m - n}.$$

COROL. I.

178. BECAUSE the Angle $MGm = \frac{a+b}{b}$

GKg , and Arches of different Circles are to one another in a Ratio compounded of the Radii and Angles that they measure: therefore

$$Gg : Mm :: KG \times GKg : MG \times \frac{a+b}{b} GKg.$$

And

And consequently also $KG \times Mm = \frac{a+b}{b} MG \times Gg$; or (which is the same thing) $KG \times Mm : MG \times Gg :: OK(a+b) : OG(b)$. Which is a constant or standing Ratio. Hence it appears, that the Dimension of the Part or Portion of the Cycloid AMD , depends on the Sum of all the Rectangles $MG \times Gg$ in the Arch GB ; which is what Mr. *Pascal* has demonstrated in common Cycloids.

Mr. *Varignon* found out this Property after a manner very different from this here.

COROL. II.

FIG. 135. 179. **W**HEN the describing Point M falls without the Circumference of the moveable Circle, there will of necessity happen one of the three following Cases. For drawing the Tangent MT , the Point of Contact G will fall (1°) in the Arch TB , as it is supposed to do in the Figure for the Investigation; and then $MC \left(\frac{2amm + 2bmm}{2am + 2bm - bn} \right)$ will be always greater than $MG(m)$. 2°, In the Point of Contact T ; and then $MC \left(\frac{2am + 2bmm}{2amm + 2bm - bn} \right) = m$, because $IG(n)$ vanishes. 3°, In the Arch TN ; and then the Value of $GI(n)$ being now negative, we shall have $MC = \frac{2amm + 2bmm}{2am + 2bm + bn}$: So that MC will be less than $MG(m)$, and always positive. Therefore in all those Cases it is evident, that the Value of (MC) the Radius of Evolution is always positive.

COROL.

COROLL. III.

180. **W**HEN the describing Point M falls FIG. 136.

within the Circumference of the moveable Circle, we have always $MC =$

$$\frac{2amm + 2bmm}{2am + 2bm - bn};$$

and it may happen that bn is greater than $2am + 2bm$, and so the Value of

(MC) the Radius of Evolution negative:

Whence it appears, that when it ceases to be positive to become negative, as it happens

when * the Point M becomes a Point of In-

flexion, then of necessity we must have $bn =$

$$2am + 2bm; \text{ and therefore } MI \times MG (mn - mm) = \frac{2amm + bmm}{b}.$$

Now if the given Line KM be called c ; from the Nature of the Circle we shall have $MI \times MG \left(\frac{2amm + bmm}{b} \right) =$

$$BM \times MN (aa - cc); \text{ and so the unknown Quantity } MG(n) = \frac{\sqrt{aab + bcc}}{2a + b}.$$

Whence if from the given Point M as a Centre, with the Distance

$$MG = \frac{\sqrt{aab - bcc}}{2a + b}$$

you describe a Circle; this will cut the moveable Circle in the Point G ,

wherein it touches the immoveable Circle serving as its Base, when the describing Point M falls in the Point of Inflexion F .

If MR be drawn perpendicular to BN , it is evident that the said $MG \left(\frac{\sqrt{aab - bcc}}{2a + b} \right)$ will

be less than $MR (\sqrt{aa - cc})$, and that it must be equal to the same when b becomes infinite,

viz. when the Base of the Cycloid becomes a right Line.

Note,

Note, That in order for the Circle described with the Radius MG to intersect the moveable Circle, it is necessary for MG to exceed MN , that is, for $\frac{\sqrt{aab-bcc}}{2a+b}$ to exceed $a-c$; and

so KM (c) exceeds $\frac{aa}{a+b}$. Whence it is manifest, that in order to have a Point of Inflexion in the Cycloid AMD , KM must be less than KN , and greater than $\frac{aa}{a+b}$.

LEMMA III.

FIG. 138. 181. **L**ET two Triangles ABb , CDd each have one Side (Bd , and Dd) infinitely small with respect to the others: I say, the Triangle ABb is to the Triangle CDd in a Ratio compounded of the Angle BAb to the Angle DCd , and of the Square of the Side AB or Ab to the Square of the Side CD or Cd .

For if from the Centres A, C , and with the Distances AB, CD the Arches BE, DF be described; it is manifest, * that the Triangles ABb, CDd do not at all differ from the Sectors of the Circles ABE, CDF . Whence, &c.

If the Sides AB, CD are equal; the Triangles ABb, CDd shall be to each other as their Angles BAb, DCd .

PROP. V.

FIG. 135. 182. **T**HE same things being supposed; it is required to square the Space $MGBA$ comprehended under the Perpendiculars MG, BA

of FLUXIONS.

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to the Cycloid, the Arch GB, and the Portion AM of the Semi-cycloid AMD granting the Quadrature of the Circle.

The Angle $GMg \left(\frac{n}{2m} GKg \right)$ is to the

Angle $MGm \left(\frac{a+b}{b} GKg \right)$, as the * lit- * Art. 181.

the Triangle MGg , whose Base is Gg the Arch of the moveable Circle, is to the little Triangle or Sector GMm ; and therefore the

$$\text{Sector } GMm = \frac{2m}{n} MGg \times \frac{a+b}{b} = \frac{2a+2b}{b}$$

$$MGg + \frac{2ap+2bp}{bn} MGg \text{ by calling } MI, p;$$

and putting $p+n$ for m . Now * the little * Art. 187.

Triangle or Sector KGg is to the little Triangle MGg in a Ratio compounded of the Square of KG to the Square of MG , and of the Angle GKg to the Angle GMg ; that is,

$$:: aa \times GKg : mm \times \frac{n}{2m} GKg. \text{ And therefore}$$

$$\text{the little Triangle } MGg = \frac{mn}{2aa} GKg. \text{ Now}$$

putting this Value in $\frac{2ap+2bp}{bn} MGg$ for the

Triangle MGg , and there will come out the

$$\text{Sector } GMm = \frac{2a+2b}{b} MGg + \frac{a+b \times pm}{aab}$$

KGg . But because of the Circle $GM \times MI (pm) = BM \times MN (cc - aa)$, which is an invariable Quantity, being so in all Situations of the describing Point M ; and consequently $GMm + MGg$ or mGg , that is, the small cyc-

$$\text{cloidal Space } GMmg = \frac{2a+2b}{b} MGg +$$

$a+b$

$\frac{a+b \times cc - aa}{aab} KGg$. Therefore because $GMm\bar{g}$

is the Fluxion of the cycloidal Space $MGBA$, and MGg the Fluxion of the circular Space MGB contained under the right Lines MG , MB , and the Arch GB , and likewise since the little Sector KGg is the Fluxion of the Sector

* Art. 96.

KGB ; it follows * that the cycloidal Space

$$MGBA = \frac{2a+3b}{b} \times MGB + \frac{a+b \times cc - aa}{aab}$$

KGB . Which was to be found.

FIG. 139.

When the describing Point M falls without the Circumference BGN of the moveable Circle, and the Point of Contact G in the

* Art. 180.

Arch NT ; it is manifest * that the Perpendiculars MG , mg intersect each other in the Point C , and then will $m = p - n$. Where-

fore the little Sector $GMm = -\frac{2a-2b}{b}$

$$MGg + \frac{2ap+2bp}{bn} MGg = -\frac{2a-2b}{b} \times$$

$$MGg + \frac{amp+bmp}{aab} KGg, \text{ by putting (as be-}$$

fore) $\frac{mn}{2aa} KGg$ for its Equal the little Triangle

MGg ; and therefore $GMm - MGg$ or mGg ,

that is, $MCm - GCg = -\frac{2a-3b}{b} \times MGg +$

$\frac{a+b \times cc - aa}{aab} KGg$, by substituting $cc - aa$ for

pm its Equal. Now if TH be supposed to be the Position of (TM) the Tangent to the moveable Circle, when the Point T thereof touches the Base in the Point T ; then it is evident that $MCm - GCg = MGT H - mgTH$,
viz.

viz. the Fluxion of the Space $MGTH$, and that MGg is the Fluxion of MGT , and likewise KGg the Fluxion of $KG T$. Therefore* * Art. 96.

the Space $MGTH = -\frac{2a-3b}{b} MGT +$

$\frac{a+b \times cc - aa}{aab} \times KG T$. But, as we have already

demonstrated, the Space $HTBA = \frac{2a+3b}{b}$

$MTB + \frac{a+b \times cc - aa}{aab} KTB$. Whence in all

Cases we have always the Space $MGBA$

$(MGTH + HTBA) = \frac{2a+3b}{b} MTB - MGT$

or $MGB + \frac{a+b \times cc - aa}{aab} KG T + KTB$ or

KGB .

Wherefore the whole Space $DNBA$ contained under (DN, BA) two Perpendiculars to the Cycloid, the Arch BGN , and the Semicircle AMD , is

$= \frac{2a+3b}{b} + \frac{a+b \times cc - aa}{aab} \times$

KGB ; because the Sector KGB , and the circular Space MGB , do each become the Semicircle KGB , when the Point of Contact G falls in the Point N .

When the describing Point M falls within the moveable Circle, we must put $aa - cc$, instead of $cc - aa$ in the foregoing Expressions, because then $BM \times MN = aa - cc$.

If you make $c = a$, we shall have the Quadrature of Cycloids, whose generating Points are in the Circumference of the moveable Circle; and if b be supposed infinite, we shall have the Quadrature of Cycloids whose Bases are right Lines.

ANOTHER SOLUTION.

FIG. 140. 183. WITH the Radius OD describe the Arch DV , and with the Diameters AV , BN , the Semicircles AEV , BSN ; and draw at pleasure from the Centre O the Arch EM between the Semicircle AEV , and the Semi-cycloid AMD , as likewise the Ordinate EP . Now it is required to find the Quadrature of the Space AEM comprehended under the Arches AE , EM , and the Portion AM of the Semi-cycloid AMD .

To do this, let there be another Arch em concentrick and infinitely near to EM , another Ordinate ep , as likewise another Oe intersecting the Arch ME continued out (if necessary) in the Point F . Now call what is variable, viz. Oe , z ; VP , u ; the Arch AE , x ; and (as before) the invariable Lines OB , b ; KB or KN , a ; KV or KA , c . Then will $Fe = z$, $Pp = u$, $OP = a + b - c + u$, $PE^2 = 2cu -$

* Art. 172. uu , the Arch $EM^* = \frac{axz}{bc}$; and therefore the

Rectangle under the Arch EM , and small right Line Fe , viz. * the little Space $EMme = \frac{axzz}{bc}$. Now because of the right-angled

Triangle OPe ; $zz = aa + 2ab + bb - 2ac - 2bc + cc + 2au + 2bu$; which thrown into Fluxions, and $zz = au + bu$. Now putting this

Value for zz in $\frac{axzz}{bc}$, and the little Space $EMme$ will be $= \frac{aaxü + abxü}{bc}$.

Now

Now if the Semi-cycloid AHT be described by the Rotation of the Semi-circle AEV along the right Line VT perpendicular to VA , and the Ordinates PE, pe , be continued out meeting the same in the Points H, b : It is manifest * that $EH \times Pp$; that is, the little Space * *Art. 172*

$EHbe$ is $= xu$; and so $EMme \left(\frac{aaxü + abxü}{bc} \right)$

: $EHbe (xu) :: aa + ab : bc$. Which is a standing Ratio. But because this is always so, let the Arch EM be where it will. Therefore the Sum of all the little Spaces $EMme$; that is, the Space AEM is to the Sum of all the little Spaces $EHbe$, that is, the Space $AEH :: aa + ab : bc$. But we have * the Quadrature * *Art. 99* of the Space AEH by means of the Quadrature of the Circle; and therefore also the Quadrature of the Space AEM sought.

This may be demonstrated without any analytical Investigation, as I have shewn in the *Acta Eruditorum* for *August*, in the Year 1695.

The Quadrature of the Space AEH , may be had otherwise than from *Art. 99*. For completing the Rectangles PQ, pq , we shall have Qq or $HR : Pp$ or $Rb :: EP : PA$ or HQ . * *Art. 181* Because * the Tangent in H is parallel to the Chord AE ; and therefore $HQ \times Qq = EP \times Pp$; that is, the small Spaces $HQq, EPpe$, are always equal to each other. Whence it follows, that the Space AHQ contained under the Perpendiculars AQ, QH , and the Portion AH of the Semi-cycloid AHT , is equal to the Space APE contained under the Perpendiculars AP, PE , and the Arch AE . Therefore the Space AEH will be equal to the Rectangle PQ minus twice the circular Space APE ; that is, to the Rectangle under PE and KA plus or minus the Rectangle under

der KP and the Arch AE , according as the Point P falls below or above the Centre. And consequently the sought Space $AEM = \frac{aa+ab}{bc} \times PE \times KA + KP \times AE$.

COROL. I.

184. **W**HEN the Point P falls in K , the Rectangle $KP \times AE$ vanishes, and the Rectangle $PE \times KA$ becomes equal to the Square of KA . Whence it appears that the Space AEM is then $= \frac{aac+abc}{b}$; and consequently it may be squared without the Quadrature of the Circle.

COROL. II.

185. **I**F the Sector AKE be added to the Space AEM , the Space $AKEM$ contained under the Radii AK , KE , the Arch EM , and the Portion AM of the Semi-cycloid AMD will be (when the Point P falls above the Centre) $= \frac{bcc+2aac+2abc-2aa u-2abu}{2bc}$

$AE + \frac{aa+ab}{bc} PE \times KA$; and therefore if you

assume $VP(u) \frac{2aac+2abc+bcc}{2aa+2ab}$ (which makes

the Value of $\frac{bcc+2aac+2abc-2aa u-2abu}{2bc}$

AE nothing,) we shall have the Space $AKEM = \frac{aa+ab}{bc} PE \times KA$. Therefore it appears still

that the Quadrature thereof is had independent on that of the Circle.

Hence

Hence it is plain, that among all the Spaces AEM and $AKEM$, there are only those two above-mentioned that can be squared.

Note, *What has been demonstrated of exterior Cycloids, extends likewise to interior Cycloids, viz. those that are generated by the Rotation of the moveable Circle along the concave Part of the immoveable one; but then the Radii KB (a), KV (c) will be negative; and so the Signs of the Terms in the foregoing Forms wherein a or c are found of odd Dimensions, must be changed.*

SCHOLIUM.

186. **T**HERE are some Curves which seem to have a Point of Inflexion, and yet have not; which I think proper to explain by an Example, because some Difficulty may arise about this Matter.

Let NDN be a geometrical Curve, the FIG. 141.

Nature whereof is expressed by $z = \frac{xx - aa}{\sqrt{2xx - aa}}$

($AP = x$, $PN = z$), wherein it is evident, 1^o, That x being $= a$, $PN (z)$ vanishes. 2^o, That when x exceeds a , the Value of z is positive; and when the same is less, the said Value is negative. 3^o, That when $x = \sqrt{\frac{1}{2}aa}$, the Value of PN is infinite. Whence it appears, that the Curve NDN extends itself on each Side the Axis, cutting the same in the Point D such, that $AD = a$; and that the Asymptote thereof is the Perpendicular BG drawn thro' the Point B so, that $AB = \sqrt{\frac{1}{2}aa}$.

Now if another Curve EDF be described of such a Nature, that drawing the Perpendicular MPN at pleasure, the Rectangle under

the Ordinate PM , and the standing Quantity AD , be equal to the correspondent Space DPN ; then if $PM=y$, it is manifest that $AD \times Rm(a\dot{y}) = NP \dot{p} n$ or $NP \times P \dot{p}$ $\left(\frac{xxx - aax}{\sqrt{2xx - aa}} \right)$; and therefore $Rm(\dot{y}) : P \dot{p}$ or $RM(\dot{x}) :: PN : AD$. Whence it follows, that the Curve EDF touches the Asymptote BG continued out in the Point E , and the Axis AP in the Point D ; and so it ought to have a Point of Inflexion in D . Yet the Value of the Radius of the Evolute of it will be

Art. 78. found $-\frac{x^2}{2aa}$, being always negative, and be-

comes equal to $-\frac{1}{2}a$ when the Point M falls in D . Whence we infer*, that the whole Curve passing thro' all the Points M , is convex next to the Axis AP ; and so has not a Point of Inflexion in D . Now to unravel this.

If you assume PM on the same Side as PN , there will be formed another Curve GDH , which will be in all respects similar to EDF , and must be a Part thereof, since its Generation is the same. This being so, we must conceive the Parts of which the whole Curve consist, not to be EDF , GDH , but EDH , GDF , which touch in the Point D ; for by this the above mentioned Difficulty is solved. For Example.

FIG. 142. Let DMG be a Curve, whose Nature is expressed by $y^4 = x^4 + aaxx - b^4$ ($AP=x$, $PM=y$). From this Equation it is manifest, that the whole Curve has two Parts EDH , GDF , opposite to each other, as the common Hyperbola; so that the Distance DD or $2AD = \sqrt{-2aa + 2\sqrt{a^4 + 4b^4}}$.

If

If b be supposed to vanish, the Distance DD FIG. 143. will vanish likewise; and therefore the two Parts EDH, GDF , will touch one another in the Point D : So that one would think the said Curve had a Point of Inflexion or Retrogression in D , according as the Parts thereof were supposed to be EDF, GDH or EDG, HDF . But this Descension will easily appear, by finding the Radius of Evolution; which will be positive always, and equal to $\frac{1}{2}a$ in the Point D , as aforesaid.

By the way, we may observe that the Quadrature of the Space DPN is dependent on that of the Hyperbola; or (which comes to the same thing) the Rectification of the Curve of the Parabola; and the Part of the Curve DMF , solves the Problem proposed by Mr. BERNOULLI, in *Tom. 2.* of the *Supplements* to the *Acta Eruditorum*, page 291. FIG. 141.





S E C T. X.

The Use of Fluxions in Geometrical Curves after a new Manner; from whence is deduced the Method of DESCARTES and HUDDE.

D E F I N. I.

FIG. 144,
145, 146.

LET ADB be a Curve such, that the Parallels KMN to the Diameter AB thereof, meet it in two Points M, N ; and conceive the intercepted Part MN or PQ to become infinitely small; then is that called the *Fluxion* of the Absciss AP or KM .

C O R O L. I.

187. WHEN the Part MN or PQ becomes infinitely small; it is evident that the Abscisses AP, AQ , will each become equal to AE , and the Points M, N , will coincide in D ; so that the Ordinate ED is the greatest or least of all the Ordinates PM, NQ like it.

C O R O L. II.

188. AMONG all the Abscisses AP , it is evident that AE only has a Fluxion, because there can be none, but in that Case where PQ becomes infinitely small.

C O R O L.

COROL. III.

189. **I**F the indeterminate Lines AP or KM be called x ; and PM or AK , (y) be invariable, it is evident that x will have two different Values, viz. KM , KN or AP , AQ . Therefore the Equation expressing the Nature of the Curve ADB must be clear'd of Surds, that so the same unknown Quantity x expressing the Roots thereof (for y is looked upon as known) may have different Values. Which must be observed hereafter.

PROP. I.

190. **T**HE Nature of a Geometrick Curve ADB being given: To determine the greatest or least Ordinate thereof.

If the Equation expressing the Nature of the Curve be thrown into Fluxions, with y as a standing Quantity, and x as a variable one; it is plain * that a new Equation will be had, * *Art. 188.* one of whose Roots x , shall express such a Value AE , that the Ordinate ED will be a Maximum or Minimum.

For Example, let $x^3 + y^3 = axy$. This thrown into Fluxions with x as variable, and y as a standing Quantity, and $3xxx = ayx$; and therefore $y = \frac{3xx}{a}$. Now if this Value of y be put for it in the Equation of the Curve $x^3 + y^3 = axy$; then will $AE(x) = a^3/2$ be such, that ED is a Maximum, as has been already shewn in *Art. 48.*

2

It

It is manifest that by this way we not only can determine the Points D , when the Ordinates ED are Perpendiculars or Tangents to the Curve ADB ; but likewise when they are oblique to the Curve, *viz.* when the Points D are those of Retrogression of the first or second Kind. So that this new Manner of considering Fluxions in geometrical Curves, is more simple and less intricate in some Cases than the first *.

• SECT. 3.

SCHOLIUM.

FIG. 146. 191. **I**N Curves that have Points of Retrogression, we may observe, that the PM^s parallel to AK , meet them in two Points M, O , just as the KM^s parallel to AP do in the Points M, N : So that AP (x) continuing the same, y has two Values PM, PO . Therefore in finding the Fluxion of the Equation of that Curve, x may be consider'd as invariable, and y as variable. Consequently if $\dot{x}$ and $\dot{y}$ be taken as variable Quantities, in throwing the aforesaid Equation into Fluxions, all the Terms affected with $\dot{x}$ on one Side, and all those with $\dot{y}$ on the other, must be equal to 0. But we must observe that $\dot{x}$ and $\dot{y}$ do here denote the Fluxions of two Ordinates issuing from the same Point, and not (as before, *Sect. 3.*) the Fluxions of two Ordinates infinitely near.

COROL.

192. **I**F after having order'd the Equation expressing the Nature of the Curve wherein there is only the unknown Quantity x variable, the same be thrown into Fluxions; it is evident,

evident, 1^o, That in doing this, we only multiply every Term by the Exponent of the Power of x , and by the Fluxion of x , and afterwards divide it by x . 2^o, That the Division by x , as well as the Multiplication by $\dot{x}$ may be omitted, since in every Term it is the same. 3^o, That the Exponents of the Powers of x are in an arithmetical Progression, the first Term thereof being the Exponent of the greatest Power, and the last 0; for the Terms that may be wanting in an Equation, we have represented by a Star.

For Example, let $x^3 * -ayx + y^3 = 0$. If every Term be multiplied by the Terms of the arithmetical Progression 3, 2, 1, 0, there will arise a new Equation $3x^3 * -ayx = 0$.

$$x^3 * -ayx + y^3 = 0.$$

$$3, 2, 1, 0.$$

$$3x^3 * -ayx * = 0.$$

Whence there comes out $y = \frac{3xx}{a}$, the same as would be found by taking the Fluxion the common way.

This being premised, instead of the arithmetical Progression 3, 2, 1, 0, you may take any other at pleasure, as $m+3, m+2, m+1, m+0$, or m , (m being any positive or negative whole Number or Fraction). For multiplying $x^3 * -ayx + y^3 = 0$ by x^m , and we shall have $x^{m+3} *, \&c. = 0$; every of the Terms of which must be multiplied by those of the Progression $m+3, m+2, m+1, m$; each by that answering thereto, to get the Fluxion thus.

$$\begin{array}{r} x^{m+3} \quad * \quad -ayx^{m+1} \quad + y^3x^m = 0. \\ m+3, \quad m+2, \quad m+1, \quad m. \\ \hline m+3x^{m+1} \quad * \quad -m+1ayx^{m+1} + my^3x^m = 0. \end{array}$$

Whence there arises $\overline{m+3x^{m+1}} - \overline{m+1ayx^{m+1}} + \overline{my^3x^m} = 0$; and dividing by x^m , there comes out $m+3x^3 - m+1ayx + my^3 = 0$, as was at first found by only multiplying the proposed Equation only by the Progression $m+3, m+2, m+1, m$.

If $m = -3$, the Progression will be $0, -1, -2, -3$; and the Equation shall be $2ayx - 3y^3 = 0$. If $m = -1$, the Progression will be $2, 1, 0, -1$, and the Equation $2x^3 - y^3 = 0$.

The Signs of all the Terms of the Progression may be changed, viz. instead of $0, -1, -2, -3$, and $2, 1, 0, -1$, we may take $0, 1, 2, 3$, and $-2, -1, 0, 1$; for doing this only alters the Signs of the Terms of the new Equation, which is to be made equal to 0. That is, instead of $2ayx - 3y^3 = 0$, $2x^3 - y^3 = 0$, we shall have $-2ayx + 3y^3, -2x^3 + y^3 = 0$.

Now it is manifest, that what we have demonstrated in the aforesaid Example, may be apply'd after the same Manner to any other. Therefore if after an Equation having two equal Roots, be duly order'd, the Terms thereof be multiplied by the Terms of any arithmetical Progression taken at pleasure, it is evident that a new Equation will be formed, one of whose Roots shall be equal to one of the Roots of the first Equation. By the same Reason, if this new Equation has two equal Roots likewise, and it be multiplied by an arithmetical Progression, we shall form a third Equation,

Equation, having one Root thereof equal to one of the equal Roots of the second Equation, and so on. So that if an Equation of three equal Roots be multiplied by the Product of two arithmetical Progressions; by that means a new Equation will be formed, having one Root equal to one of the three equal Roots of the first Equation: And, in like manner, if the Equation has four equal Roots, it should have been multiplied by the Product of three arithmetical Progressions; if five, by the Product of four, &c.

This is what the Method of Mr. HUDDLE precisely consists in.

PROP. II.

193. **T**O draw a Tangent THM from a given Point T, in the Diameter AB; or a given Point H in AH, parallel to the Ordinates. FIG. 147.

From the Point of Contact M, draw the Ordinate MP, and call AT, s ; AH, t ; (one or the other of which is given) and the unknown Lines AP, x ; PM, y ; then because of the Similiarity of the Triangles TAH, TPM,

$$y = \frac{st + tx}{s}, x = \frac{sy - st}{t}; \text{ and putting these Values for } y \text{ or } x \text{ in the given Equation expressing the Nature of the Curve AMD, we shall have a new Equation freed from } x \text{ or } y.$$

Now if a right Line TD be drawn cutting the right Line AH in G, and the Curve AMD in two Points N, D, from which are let fall the Ordinates NQ, DB; it is manifest that when t expresses AG in the foregoing Equation,

tion,

tion, x or y will have two Values AQ , AB , or NQ , DB , which become equal to one another, viz. to AP or PM sought when t expresses AH ; that is, when the Secant TDN becomes the Tangent TM . Whence it follows, that that Equation must have two equal Roots. And so we will multiply it by any arithmetical Progression at pleasure; which must be repeated, if necessary, by multiplying *de Nouveau*, the same Equation by any other arithmetical Progression; that so comparing the Equations arising therefrom, we may find one of them affected with either of the unknown Quantities x or y , and having one of the given Quantities s or t in it. The following Example will be sufficient for explaining this.

EXAMPLE.

194. **L**ET $ax = yy$ express the Nature of the Curve AMD . If instead of x we put $\frac{sy-st}{t}$; then will tyy , &c. Which must have two equal Roots.

$$tyy - asy + ast = 0.$$

$$1, \quad 0, \quad -1.$$

$$tyy \quad * \quad -ast = 0.$$

Therefore multiplying orderly (as you see here) these Terms by those of the arithmetical Progression $1, 0, -1$, we shall have $as = yy = ax$; and consequently $AP(x) = s$. Whence taking $AP = AT$, and drawing the Ordinate PM , the Line TM will touch the Curve in M . But if $AH(t)$ be given instead of $AT(s)$, we must multiply the same Equation tyy , &c. by

by this other Progression 0, 1, 2, and the sought Tangent $PM(y)$ will be $= 2t$.

The same Construction may be found by putting $\frac{st+tx}{s}$ for y , in $ax=yy$. For there arises txx , &c. the Terms whereof multiplied by 1, 0, -1, produce $xx=ss$, and consequently $AP(x)=s$.

COROL.

195. **N**ow if you suppose the Point of Contact M to be given, and the Point T or H , wherein the Tangent MT intersects the Diameter AB or the Parallel AH to the Ordinates, be sought; you need only, (in the latter Equation expressing the unknown Quantity x or y with respect to the given one s or t ,) look upon this last as the unknown Quantity, and x or y as known.

PROP. III.

196. **T**HE Nature of the Geometrick Curve FIG. 148. AFD being given: To determine its Point of Inflexion F .

Draw the Ordinate FE from the Point F sought, as likewise the Tangent FL , and thro' the Point A (the Origin of the x 's) the Line AK parallel to the Ordinates. Likewise call what are unknown, viz. LA, s ; AK, t ; AE, x ; EF, y . Then because of the similar Triangles $LAK, LEF, y = \frac{st+tx}{s}$, and $x = \frac{sy-st}{t}$;

so that if these Values be put in the Equation of the Curve for y or x , we shall get a new Equation

Equation freed from x or y , as in the last Proposition.

- Now if a right Line TD be drawn intersecting the right Line AK in H , which touches the Curve AFD in M , and the Absciss in D , from which are let fall the Ordinates MP , DB ; it is evident, 1^o, That when s expresses AT ; and t , AH ; the Equation found as afore-
- *Art. 193.* said, must have two equal Roots, *viz.* * each equal to AP or PM , according as y or x be made to vanish, and another Root AB , or BD . 2^o, That when s expresses AL ; and t , AK ; the Point of Contact M coincides with the Point of Intersection D in the Point F
- *Art. 67.* sought: Because * the Tangent LF must both touch and cut the Curve in the Point of Inflection F ; and so AP , AB , the Values of x , or PM , PD , the Values of y become equal to one another, *viz.* equal to AE or EF sought. Whence the said Equation must have three equal Roots. Consequently it must be multiplied by the Product of two arithmetical Progressions at pleasure; which must be again repeated, if necessary, by multiplying it in like manner with another Product of any two arithmetical Progressions; that so by comparing those Equations resulting therefrom, the unknown Quantities s and t may vanish.

EXAMPLE.

197. **L**ET $ayy = xyy + aax$ express the Nature of the Curve AFD . If $\frac{sy - st}{t}$ be put for x , there will arise $sy^3 - styy - tyy$, &c.

$$sy^3 =$$

$$sy^3 - styy + aasy - aast = 0.$$

— at

$$1, \quad 0, \quad -1, \quad -2.$$

$$3, \quad 2, \quad 1, \quad 0.$$

$$3sy^3 * - aasy * = 0.$$

Which being multiplied by 3, 0, —1, 0, the Product of two arithmetical Progressions 1, 0, —1, —2, and 3, 2, 1, 0, gives us $yy = \frac{1}{3}aa$; and putting this Value in the Equation of the Curve, the unknown Quantity $AE (x)$ will be $= \frac{1}{4}a$.

ANOTHER SOLUTION.

198. **T**HE aforesaid Problem may be solved likewise, in considering that but only one Tangent LF or KF can be drawn from the same Point L or K , since it outwardly touches the concave Part AF , and inwardly the convex Part FD ; whereas from any other Point T or H , taken in AL or AK between A and L , or A and K , we can draw two Tangents TM , TD or HM , HD , the one to the concave, and the other to the convex Part: So that the Point of Inflexion (F) may be conceived as the Point of Coincision of the two Points of Contact M and D . If then $AT (s)$ or $AH (t)$ be supposed to be given, and you seek * the Value of x or y with respect to s or t ; we shall get an Equation having two Roots AP , AB , or PM , BD , which will be each equal to the sought Quantity AE or EF , when s expresses AL , and t , AK . Therefore that Equation must be multiplied by any arithmetical Progression, &c.

FIG. 149,
150.

* Art. 194.

Q

EXAMPLE

EXAMPLE.

199 **L**ET as before, $ayy = xyy + aax$; then again will $sy^3 - sty - aty + aasy - aast = 0$, which being multiplied by the arithmetical Progression 1, 0, -1, -2, and there arises $y^3 - 2ay - 2aat = 0$, freed from s , having two unequal Roots, viz. PM, BD , when t expresses AH , and two equal ones each to EF sought, when t expresses AK . Therefore multiplying *de nouveau* this latter Equation by the arithmetical Progression 3, 2, 1, 0, and there will arise $3yy - aa = 0$; and therefore $EF(y) = \sqrt{\frac{1}{3}aa}$. Which was to be found.

PROP. IV.

FIG. 151. 200. **F**ROM a given Point C without a Curve Line AMD , to draw CM perpendicular to that Curve.

Draw MP, CK , perpendicular to the Diameter AB , and with the Distance CM describe a Circle from the Centre C ; then it is manifest that it shall touch the Curve AMD in the Point M . Now calling the known Lines AP, x ; PM, y ; CM, r ; and the unknown ones AK, s ; KC, t ; and we shall have PK or $CE = s - x$, $ME = y + t$; and because of the right-angled Triangle MEC , $y = -t + \sqrt{rr - ss + 2sz - xx}$, $x = s - \sqrt{rr - tt - 2ty - yy}$: So that putting these Values for y or x in the Equation of the Curve, we shall get a new Equation freed from y or x .

Now if another Circle be described from the Centre C , cutting the Curve in two Points N, D ,

N, D , from which the Perpendiculars NQ , DB , are let fall: It is manifest that when r expresses the Radius CN , or CD , in the Equation aforegoing, x or y will have two Values AQ , AB or NQ , DB , which become equal to one another (*viz.*) to the Quantity AP or PM sought, when r expresses the Radius CM . Whence that Equation must have two equal Roots; therefore it must be multiplied, &c.

EXAMPLE.

201. **L**ET $ax=yy$ express the Nature of the Curve AMD , in which putting $s=\sqrt{rr-tt-2ty-yy}$ for its Equal x , and then $as-yy=a\sqrt{rr-tt-2ty-yy}$: So that squaring each Side, and afterwards duly ordering the Equation, we shall get y^4 , &c. which must have two equal Roots when y expresses PM sought.

$$\begin{array}{r}
 y^4 \quad * - 2asyy + 2aaty + aass = 0. \\
 \quad \quad + aa \quad \quad \quad - aarr \\
 \quad \quad \quad \quad \quad - aatt \\
 \hline
 4y^4 \quad * - 4asyy + 2aaty \quad * = 0. \\
 \quad \quad + 2aa
 \end{array}$$

Therefore multiplying it by the arithmetical Progression 4, 3, 2, 1, 0, (as here you see it done) and there comes out $4y^5 - 4asy + 2aay + 2aat = 0$; and the Value of y will be $= PM$ sought.

If the given Point C falls in the Diameter AB , then will $t=0$, and consequently all the Terms affected with it will go out. Whence $4as - 2aa = 4yy = 4ax$, by putting yy instead of Q^2

FIG. 152.

of

of ax , which is equal to it. Therefore we get $x = s - \frac{1}{2}a$; that is, if CP be assumed equal to $\frac{1}{2}$ the Parameter, and the Ordinate PM be drawn perpendicular to AB , and the right Line CM be drawn, it will be perpendicular to the Curve AMD .

C O R O L.

FIG. 152. 202. **N**ow if the Point M be given, and the Point C be supposed to be that sought; then in the last Equation expressing the Value of AC (s) with respect to AP (x) or PM (y), we must esteem these latter as known, and the other as unknown.

P R O P. V.

FIG. 153. 203. **A**NY Point M being given in a Curve AMD , whose Nature is also given: To find MC the Radius of the Curvature in that Point: Or to find the Point C , being the Centre of a Circle of equal Curvature with the Curve in the Point M .

Draw MP , CK , perpendicular to the Axis; denote the Lines by the same Letters as in the Problem foregoing. Then we shall get the same Equation as there, wherein we must observe that the Letter x or y does here denote a given Magnitude, tho' it did there an unknown one; and on the contrary s , t , esteemed as known ones there, are in Reality here unknown as well as r .

Now it is manifest, 1^o, That C the Centre of the Circle of equal Curvature with the Curve in the Point M , must be in MG perpendicular

pendicular to the Curve. 2°, That a Circle can always be described which will touch the Curve in M , and at least cut it in two Points (the nearest of which is D , from which the Perpendicular DB is let fall); because we can always find a Circle that will intersect any Curve Line, except a Circle, in four Points at least, and the Point of Contact M is equivalent but to two Intersections. 3°, That the nearer the Centre G thereof is to C the extreme of the Radius sought, the nearer will the Point of Intersection D be to the Point of Contact M . So that when the Point G falls in C , the Point D falls in M ; because * the * *Art. 76.* Circle described with the Radius CM sought, must both touch and cut the Curve in the same Point M . Whence it is manifest, that s expressing AF , and t , FG , the Equation must have two equal Roots, (*viz.* * each equal * *Art. 200.* to AP or PM according as y or x are made to vanish) and another AB or BD , which becomes likewise equal to AP or PM , when s and t express AK , KC , sought; and so the said Equation must have three equal Roots.

EXAMPLE.

204. **L**ET $ax = yy$ express the Nature of the Curve AMD . We shall get * y^4 , &c. * *Art. 201.* which being multiplied by 8, 3, 0, —1, 0, the Product of two arithmetical Progressions 4, 3, 2, 1, 0, and 2, 1, 0, —1—2, and there arises $8y^4 = 2aaty$. See the Operation.

Q 3

y^4

$$\begin{array}{r}
 y^4 * - 2asyy + 2aaty + aass = 0. \\
 + aa \quad \quad - aarr \\
 \quad \quad \quad + aatt \\
 \hline
 4, \quad 3, \quad 2, \quad 1, \quad 0. \\
 2, \quad 1, \quad 0, \quad -1, \quad -2. \\
 \hline
 8y^4 * * - 2aaty * = 0.
 \end{array}$$

Whence KC or $PE(t) = \frac{4y^3}{aa}$.

If it be required to find an Equation expressing the Nature of the Curve passing thro' all the Points (C), we must still multiply y^4 , &c. by 0, 3, 4, 3, 0, the Product of two arithmetical Progressions 4, 3, 2, 1, 0, and 0, 1, 2, 3, 4; and then will $8asy - 4aay = 6aat$: Whence (making, for Brevity's sake, $s - \frac{1}{2}a = u$) there comes out $y = \frac{3at}{4u}$, and $4y^3 = \frac{27a^3t^3}{16u^3} = aat$; and therefore $16u^3 = 27att$. Consequently the Curve passing thro' all the Points (C) is a second cubick Parabola, the Parameter to the Axis being $= \frac{27a}{16}$, and the Vertex is distant from the Vertex of the given Parabola by $\frac{1}{2}a$, since $u = s - \frac{1}{2}a$.

When the Position of the Parts of the Curve adjoining to the given Point M , is alike on each side that Point, as it happens when the Curvature there is a *Maximum* or *Minimum*: Then one of the Intersections of the touching Circle cannot coincide with the Point of Contact, except the other does so likewise; so that the Equation must have four equal Roots. Now if y^4 , &c. be multiplied by 24, 6, 0, 0, 0, the Product of three arithmetical Progressions 4, 3, 2, 1, 0, and 3, 2, 1, 0, -1, and 2, 1, 0, -1, -2; we shall get $24y^4 = 0$. Therefore

the Point *M* must fall in *A* the Vertex of the Parabola, in order for the Parts of the Curve adjoining to it on each Side to be similar or alike.

ANOTHER SOLUTION.

205. **I**F you call to mind what has been demonstrated in *Art.* 76, viz. that but one Line *CM* can be drawn from the Point *C* sought perpendicular to the Curve *AMD*; whereas from any [other Point *G* in the said Perpendicular, there can be drawn two Perpendiculars *MG*, *GD*, to the Curve. From this Consideration we can solve the Problem thus: Suppose the Point *G* to be given, and seek *the Value of *x* or *y* with regard to *s* and *t* which are given: Then it is plain that the Equation must have two unequal Roots, viz. *AP*, *AB*, or *PM*, *BD*, which become equal when the Point *G* coincides with the Point *C* sought. Wherefore multiply that Equation by any arithmetical Progression, &c.

FIG. 154.

**Art.* 200.

EXAMPLE.

206. **L**ET, as before, $ax=yy$, then will * $4y^3$, **Art.* 101. &c. See the Operation.

$$\begin{array}{r}
 4y^3 \quad * \quad - \quad 4asy + 2aat = 0. \\
 \quad \quad \quad + \quad 2aa \\
 \hline
 2, \quad 1, \quad 0, \quad -1. \\
 8y^3 \quad + \quad * \quad - 2aat = 0.
 \end{array}$$

Which being multiplied by the arithmetical Progression 2, 1, 0, -1, and there comes out,

as * before, $t = \frac{4y^3}{aa}$.

**Art.* 204.

Q4

COROL.

COROL.

- FIG. 153, 207. **I**T is manifest that the Point wherein
 154. the Radius of the Curvature touches
 *Art. 203. the Curve, may be consider'd * as the Place
 where the Point wherein the Circle of the
 same Curvature with the Curve touches it,
 coincides with the Point of Intersection of the
 *Art. 205. said Circle; or else as * the Point wherein
 two Points of Contact of different concentric
 Circles coincide. Just as a Point of
 *Art. 196. Inflection is looked upon *, as that wherein
 a Points of Contact coincides with the In-
 *Art. 198. tersection of the same right Line; or * as the
 Coincision of two Points of Contact of two
 right Lines issuing from the same Point.

PROP. VI.

- FIG. 155. 208. **T**O find an Equation expressing the Nature of the Caustick $AFGK$, generated in the Quadrant $CAMNB$, by the reflected Rays MH, NL , &c. the incident Rays PM, QN , &c. being all parallel to CB .

We may observe, 1°, That if the reflected Rays MF, NG , touching the Caustick in F, G , be continued out to meet the Radius CB in the Points H, L ; then will $MH = CH$, and $NL = CL$. For the Angle $CMH = CMP = MCH$; and in like manner the Angle $CNL = CNQ = NCL$.

2°, That from a given Point F in the Caustick AFK , there can be drawn but one right Line MH equal to CH ; whereas from a given Point D between the Quadrant AMB , and the

the Caustick AFK , two Lines MH , NL , may be drawn so, that $MH=CH$, and $NL=CL$. For but one Tangent MH can be drawn from the Point F ; but from D two Tangents MH , NL , can be drawn. This being well understood:

It is required to draw the right Line MH from a given Point D , in such manner, that it be equal to the Part CH determined thereby in the Radius CB .

Draw MP , DO , parallel to CB , and MS parallel to CA ; call what is given, *viz.* CO or RS , u ; OD , z ; AC or CE , a ; and the unknown Quantities CP or MS , x ; PM or CS , y ; CH or MH , r . Then because of the right-angled Triangle MSH , $rr=rr-2ry+yy+xx$; and so $CH(r)=\frac{xx+yy}{2y}$. Moreo-

ver, because of the similar Triangles MRD , MSH , $MR(x-u):MS(x)::RD(z-y):SH=\frac{zx-xy}{x-u}$. And therefore $CS+SH$ or

$$CH=\frac{zx-xy}{x-u}=\frac{xx+yy}{2y}=\frac{aa}{2y}, \text{ by putting } aa$$

for $xx+yy$. Whence (multiplying crosswise) and there arises $aa x - a a u = 2 z x y - 2 u y y$; and putting $aa - x x$ for yy , there comes out $2 z x y = a a x + a a u - 2 u x x$. Then squaring both Sides, to get rid of the Surds, and again substituting $aa - x x$ for yy , and at length we have $4 u u x^4 - 4 a a u x^3 - 4 a a u u x x + 2 a^4 u x + a^4 u u = 0$.

$$4 z z$$

$$- 4 a a z z$$

$$+ a^4$$

Now when u expresses CO , and z OD , it is manifest that this Equation must have two

unequal Roots, viz. CP, CQ ; and on the contrary, when u expresses CE ; and z, EF ; CQ will become equal to CP ; so that then it will have two equal Roots. Therefore if the Terms thereof be multiplied by the Terms of the Arithmetical Progressions, 4, 3, 2, 1, 0, and 0, 1, 2, 3, 4, there will be two new Equations formed, from whence there will arise this Equation, after the unknown Quantity x is gotten out.

$$64z^6 - 48aaz^4 + 12a^4zz - a^6 = 0, \\ + 192uu - 96aauu - 15a^4uu \\ + 192u^4 - 48aa u^4 \\ + 64u^6$$

expressing the Relation between the Absciss GE (u) and the Ordinate EF (z). Which was to be found.

The touching Point F may be determined by what is explained in the eighth Section. For if another incident Ray pm be conceived infinitely near PM , it is plain that the reflected Ray mb shall cut MH in the Point F sought; from which having drawn FE parallel to PM , and making $CE = u$, $EF = z$, $CP = x$, $PM = y$, $CM = a$, you will find as before $\frac{aax + aa u - 2uxx}{xy} = 2z$. Now it is manifest,

that CM, CE, EF continue the same while CP and PM vary. Therefore with a, u, z , as standing Quantities, and x and y variable ones, throw that Equation into Fluxions, which will be $2uyxxx + aa uy\dot{x} - aaxx\dot{y} - aaux\dot{y} + 2ux^2\dot{y} = 0$. Wherein substitute $-\frac{yy}{x}$ for its Equal $\dot{x}$, (because they are such by throwing $yy = aa - xx$ into Fluxions) and then $aa - xx$ for yy , and

and at length there comes out $CE(u) =$

$$\frac{x^3}{aa}$$

If the Curve AMB be not a Quadrant, but any other Curve, whereof the right Line MC is the Radius of Evolution in the Point M ; it is manifest, * that the Portion Mm * *Art. 76.* thereof may be esteemed as an Arch of a Circle described about the Centre C . Therefore if from that Centre the right Line CP be drawn perpendicular to the incident Ray PM , and

assuming $CE = \frac{x^3}{aa}$ ($CP = x$, $CM = a$), you

draw EF parallel to PM ; it shall cut the reflected Ray MH in the Point F , wherein it touches the Caustick AFK .

If thro' all the Points M, m in any Curve AMB be drawn straight Lines MC, mC to a given Point C in the Axis AC of it, and other right Lines MH, mb terminated by CB perpendicular to the Axis; in such manner that the Angle $CMH = MCH$, and $Cmb = mCb$; and it be requir'd to find the Point F , in every Line MH wherein it touches the Curve AFK formed by the continual Interfection of the said right Lines MH, mb , we shall find as before

$$CH = \frac{xx + yy}{2y} = \frac{zx - uy}{x - u} : \text{From whence we}$$

$$\text{get } \frac{x^3 + uyy + xyy - uxx}{xy} = 2z, \text{ which thrown}$$

into Fluxions (with u and z as standing Quantities, and x and y as variable ones) will be $2x^3y\dot{x} - uxx\dot{y}\dot{x} - uxx\dot{y}\dot{x} - x^4\dot{y} + ux^3\dot{y} + xxy\dot{y}\dot{y} + uxy\dot{y}\dot{y} - uy^3\dot{x} = 0$; and therefore $CE(u) =$

$$\frac{2x^3y\dot{x} - x^4\dot{y} + xxy\dot{y}\dot{y}}{xxy\dot{x} - x^3\dot{y} + y^3\dot{x} - xyy\dot{y}} \quad \text{Now the Nature of}$$

the

the Line AMB being given, we shall have a Value of y in x , which being substituted in the Expression of CE , and the same will be freed from Fluxions.

PROP. VII.

FIG. 156. 209. **L**ET AO be an indefinite right Line, the Beginning whereof is the stable Point A ; and let there be an infinite Number of Parabola's BFD, CDG , having the right Line AO as a common Axis, to which the right Lines AB, AC intercepted between the stable Point A and their Vertices B and C , are the Parameters. It is requir'd to find the Nature of the Line AFG touching all those Parabola's.

We may observe, 1^o, That any two of these Parabola's BFD, CDG intersect one another in the Point D , situate between the Line AFG and the Axis AO ; and that when $AC = AB$, the Point of Intersection D coincides with the Point of Contact F . This being well understood,

It is required to draw a Parabola thro' the given Point D having the denoted Property. Draw the Ordinate DO , and call the given Quantities AO, u ; OD, z ; and the unknown one AB, x ; then by the Nature of the Parabola given $AB \times BO (ux - xx) = DO^2 (zz)$; and ordering the Equation $xx - ux + zz = 0$. Now when u expresses AO , and z, OD ; the said Equation will have two unequal Roots, viz. AB, CA : and on the contrary when u expresses AE ; and z, EF ; AC becomes equal to AB , that is, the Equation then has two equal Roots. Therefore it must be multiply'd
by

by the Arithmetical Progression 1, 0, -1: and so $x = z$, and substituting z for x , there arises $u = 2z$, which must express the Nature of the Line AFG . Whence it follows, that AFG is a right Line making the Angle FAO with AO , being such that AE is the Double of EF .

If the Problem is requir'd to be solved generally, viz. let the Parabola's BFD , CDG be of what nature you please. Recourse is to be had to the Method explain'd in the Eighth Section, and the same must be used thus. Call AE, u ; EF, z ; AB, x ; then will

$\overline{u-x}^m \times x^n = z^{m+n}$ express generally the Nature of the Parabola BF . This Equation thrown into Fluxions (making x and z invariable, and x variable) and we have

$$-m \times \overline{u-x}^{m-1} \dot{x} \times x^n + n x^{n-1} \dot{x} \times \overline{u-x}^m = 0;$$

and dividing by $\overline{u-x}^{m-1} \dot{x} \times x^{n-1}$, there

comes out $x = \frac{n}{m+n} u$; and therefore $u-x =$

$\frac{m}{m+n} u$. Now substituting these Values for

$u-x$, and x in the general Equation; and

making (for brevity's sake) $\frac{m}{m+n} = p$, $\frac{n}{m+n}$

$= q$, $m+n = r$, and then will z be $= \sqrt[r]{p^m q^n}$.

Whence it appears that AFG is always a right Line, be the Parabola's of what Nature soever, the Ratio of AE to EF only varying.

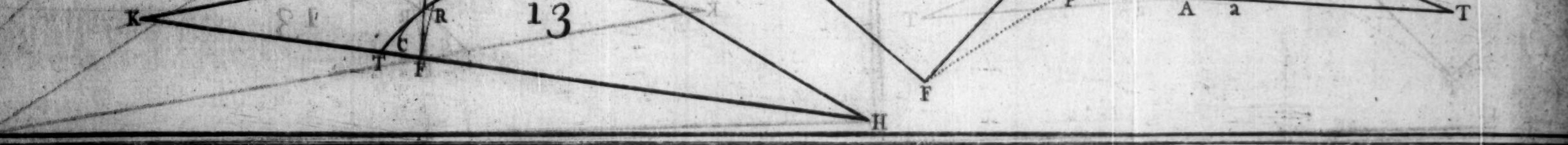
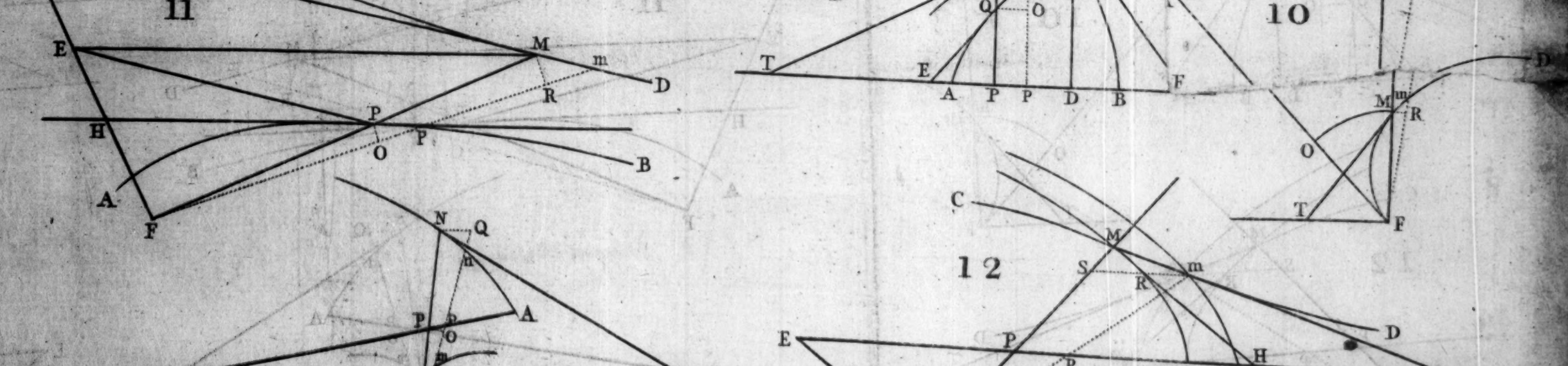
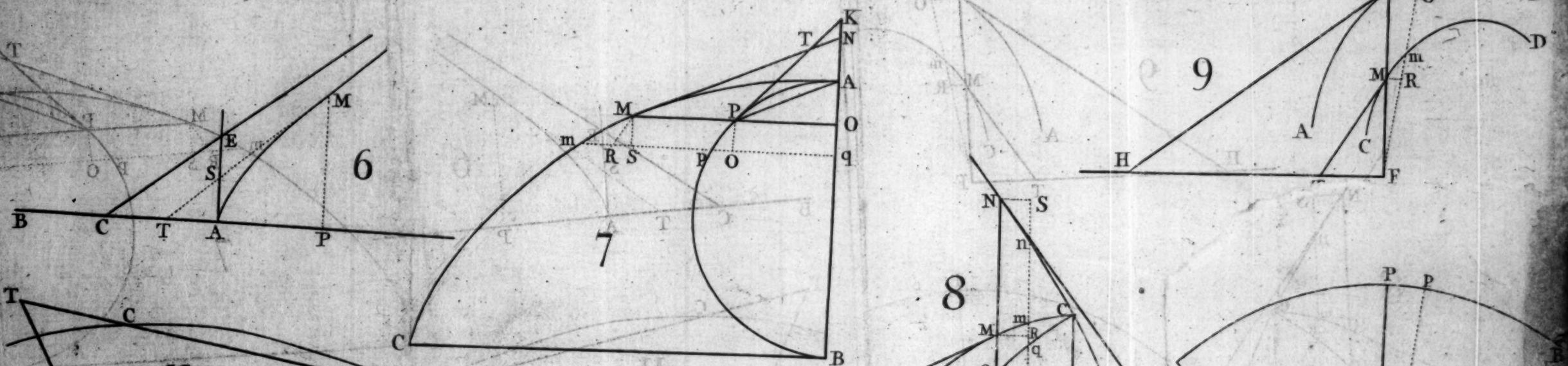
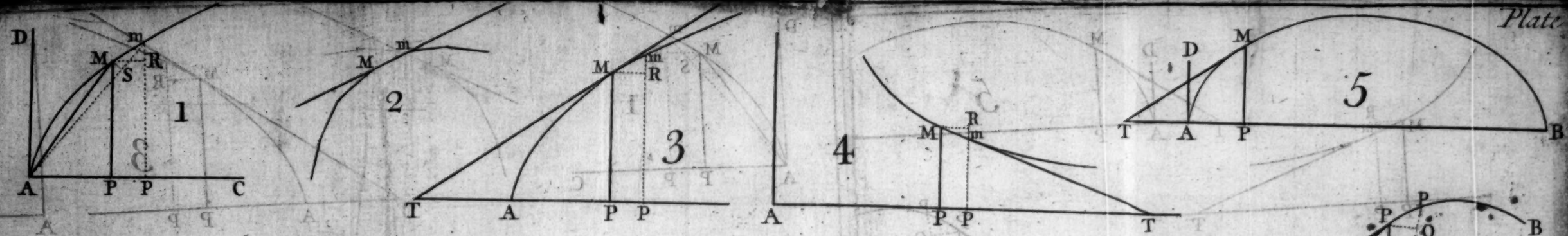
From what has been explained in this Section, it evidently appears how Descartes and Hudde's Method must be used in the Solution of Problems of these Kinds when the Curves are geometrical.

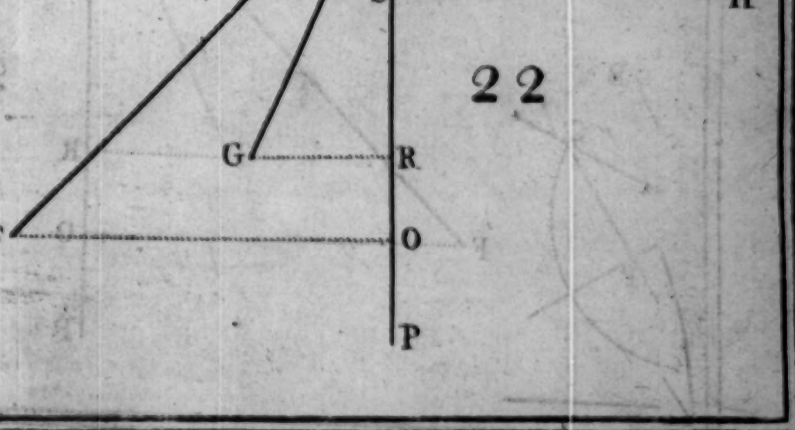
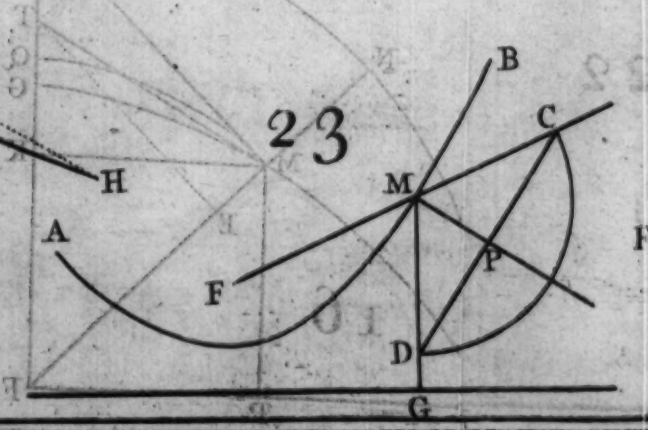
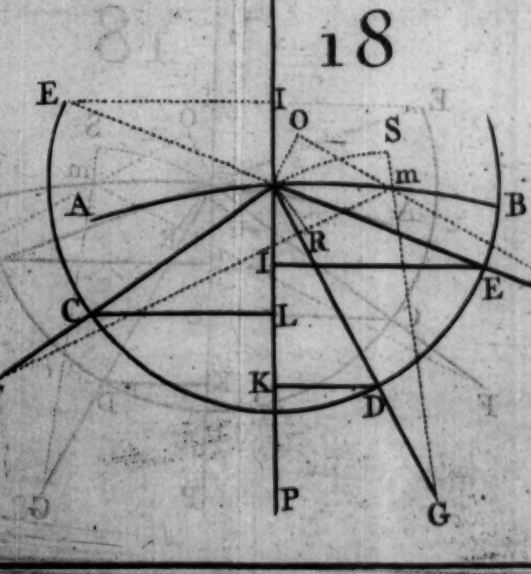
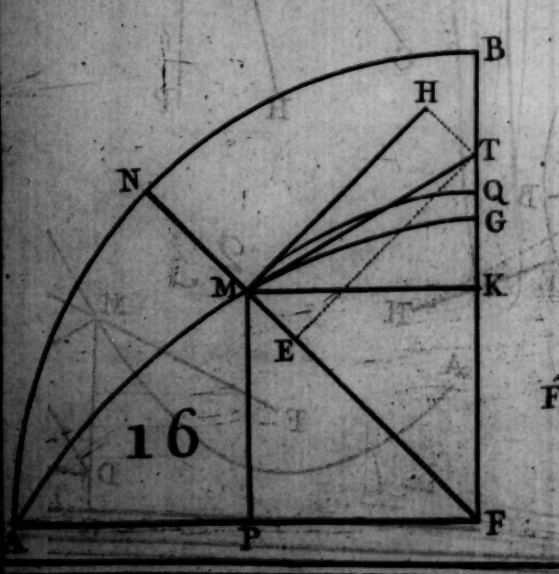
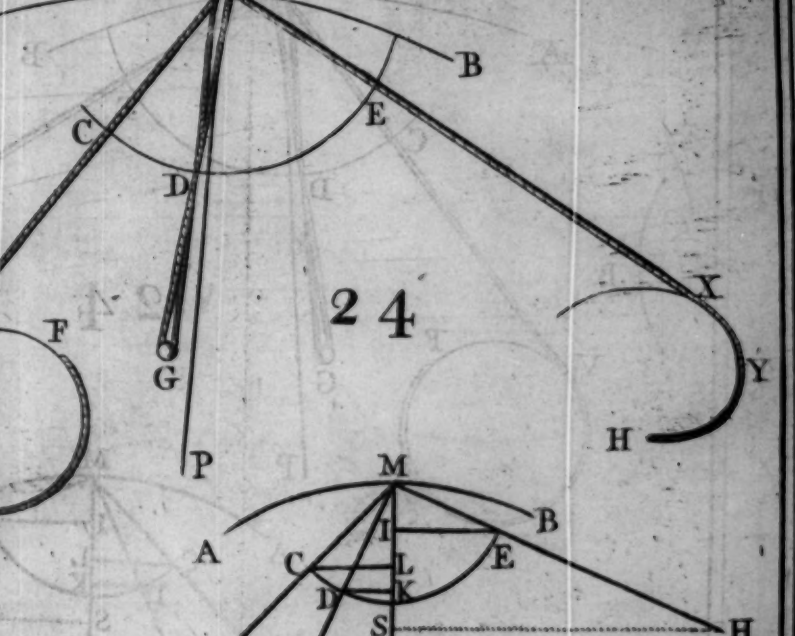
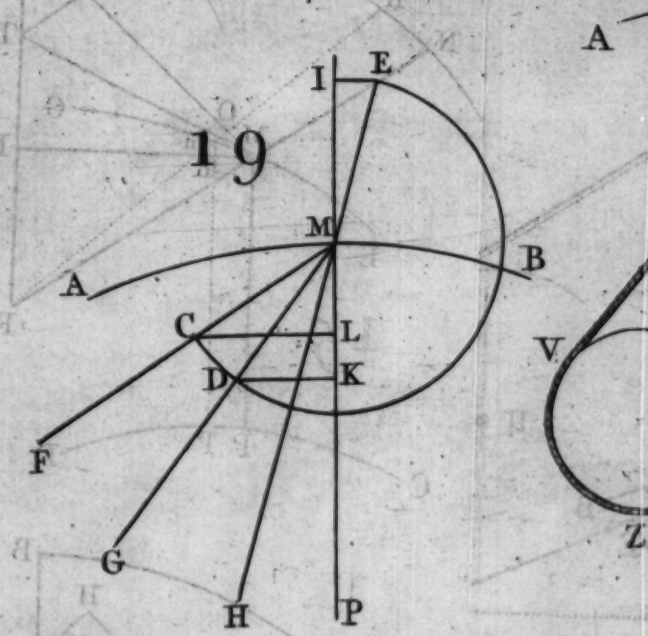
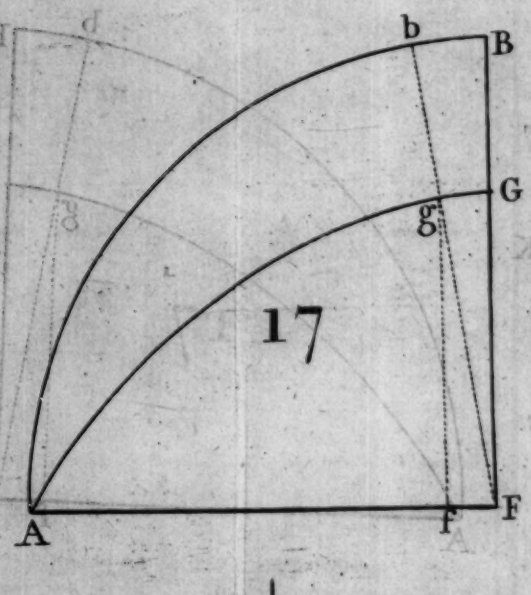
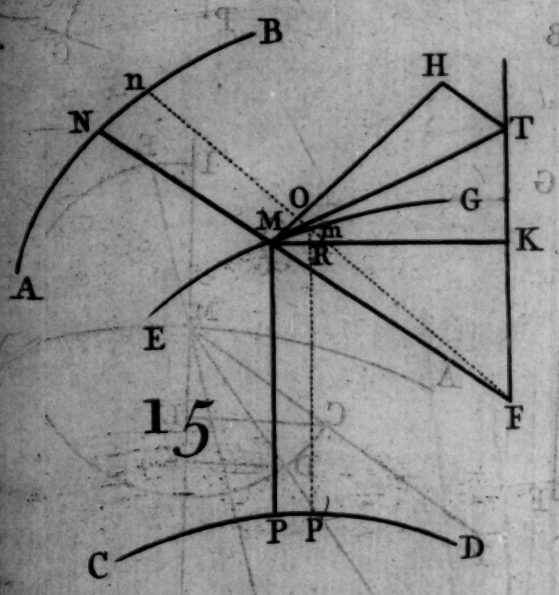
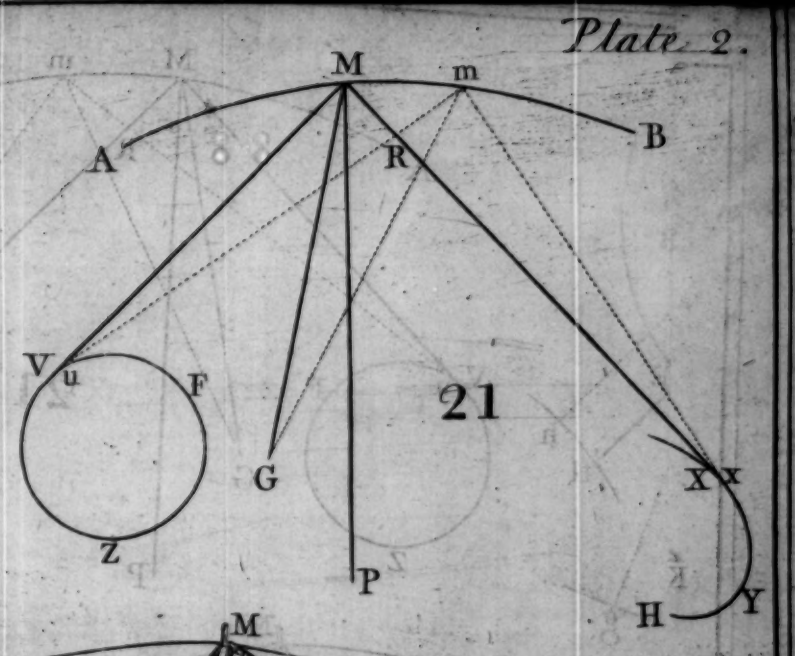
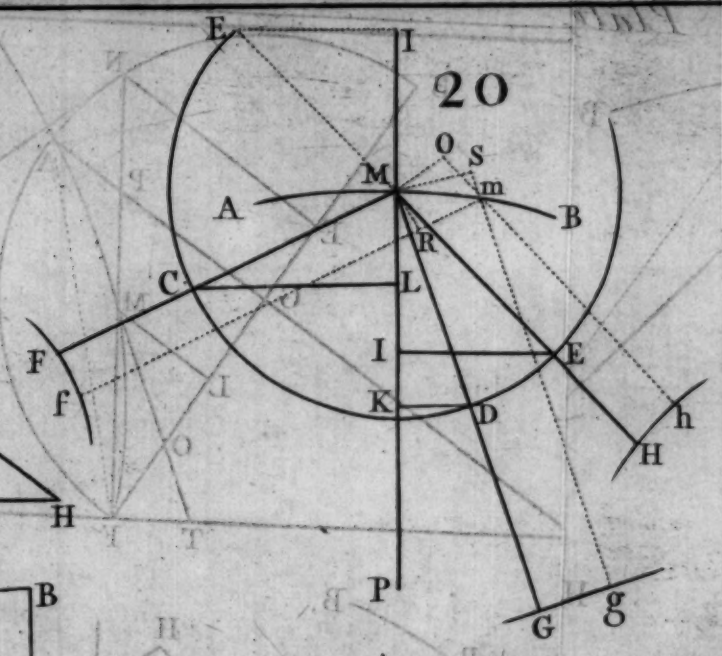
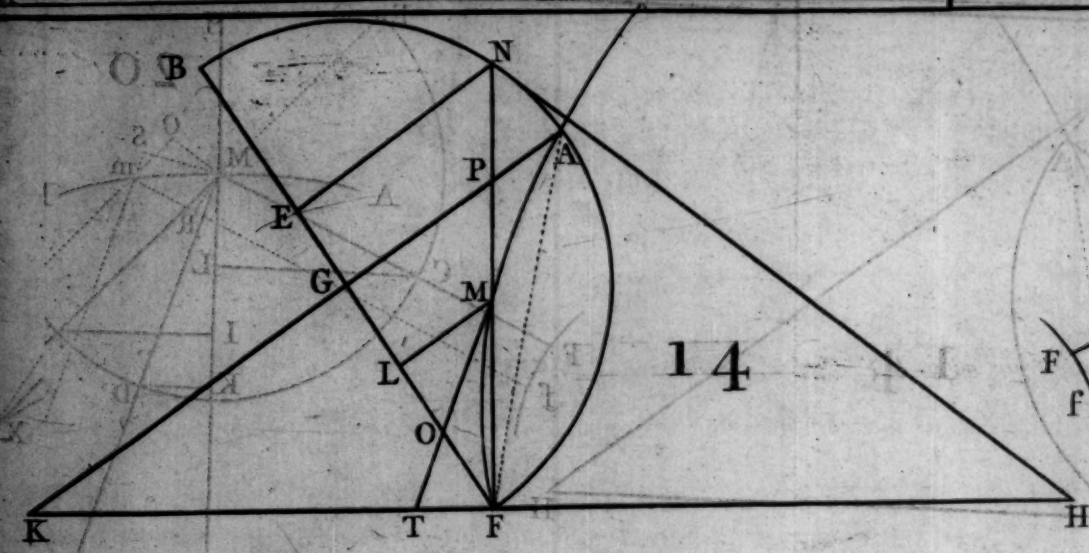
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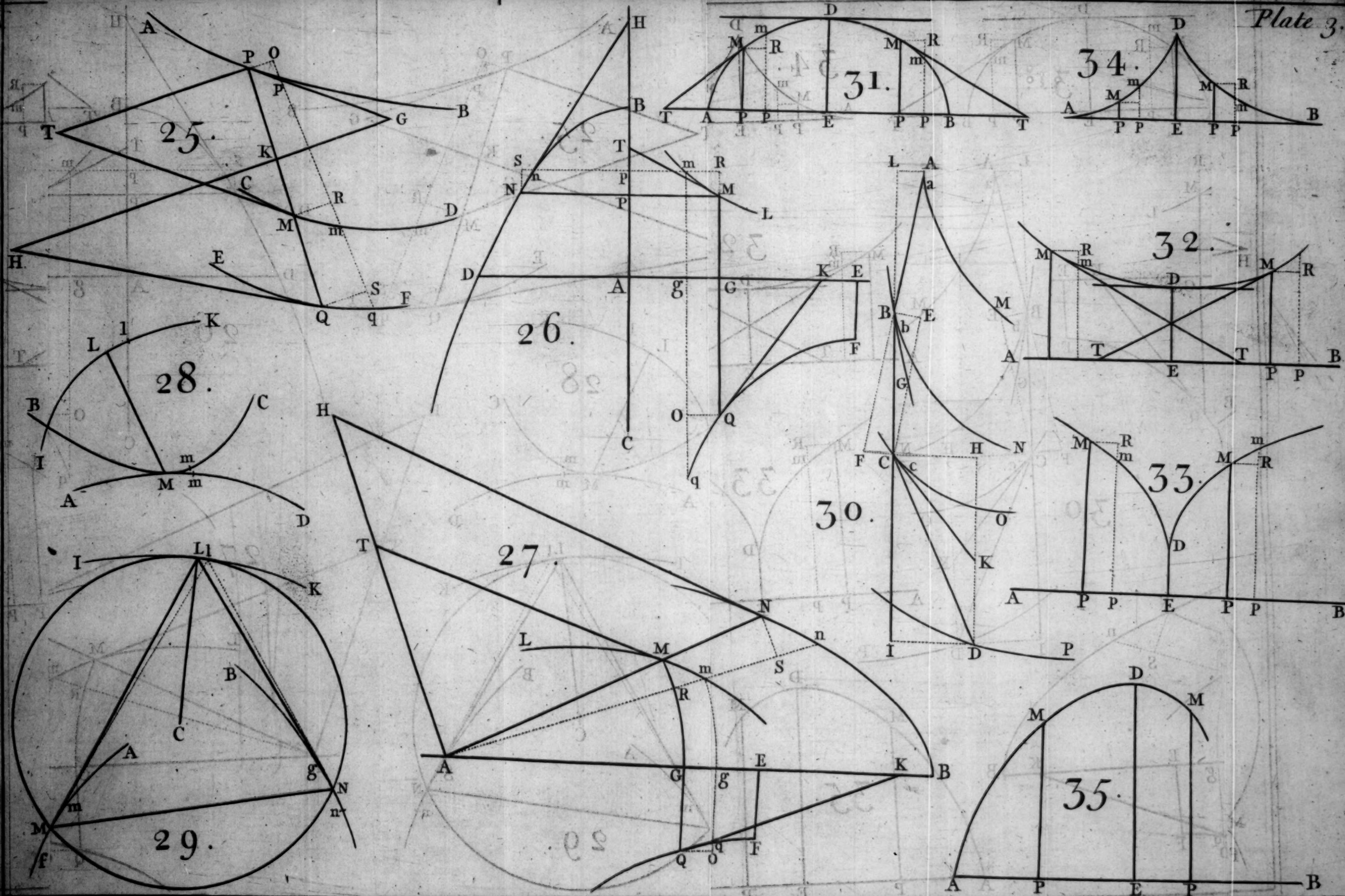
But it is not comparable to the Method of Fluxions, which furnishes us with general Solutions, and extends to all Kinds of Curves, without any Necessity of clearing Equations of Surds. Whereas, by the former Method, we only get particular Solutions, and are necessitated to throw out the Surds: which very often cannot be done.

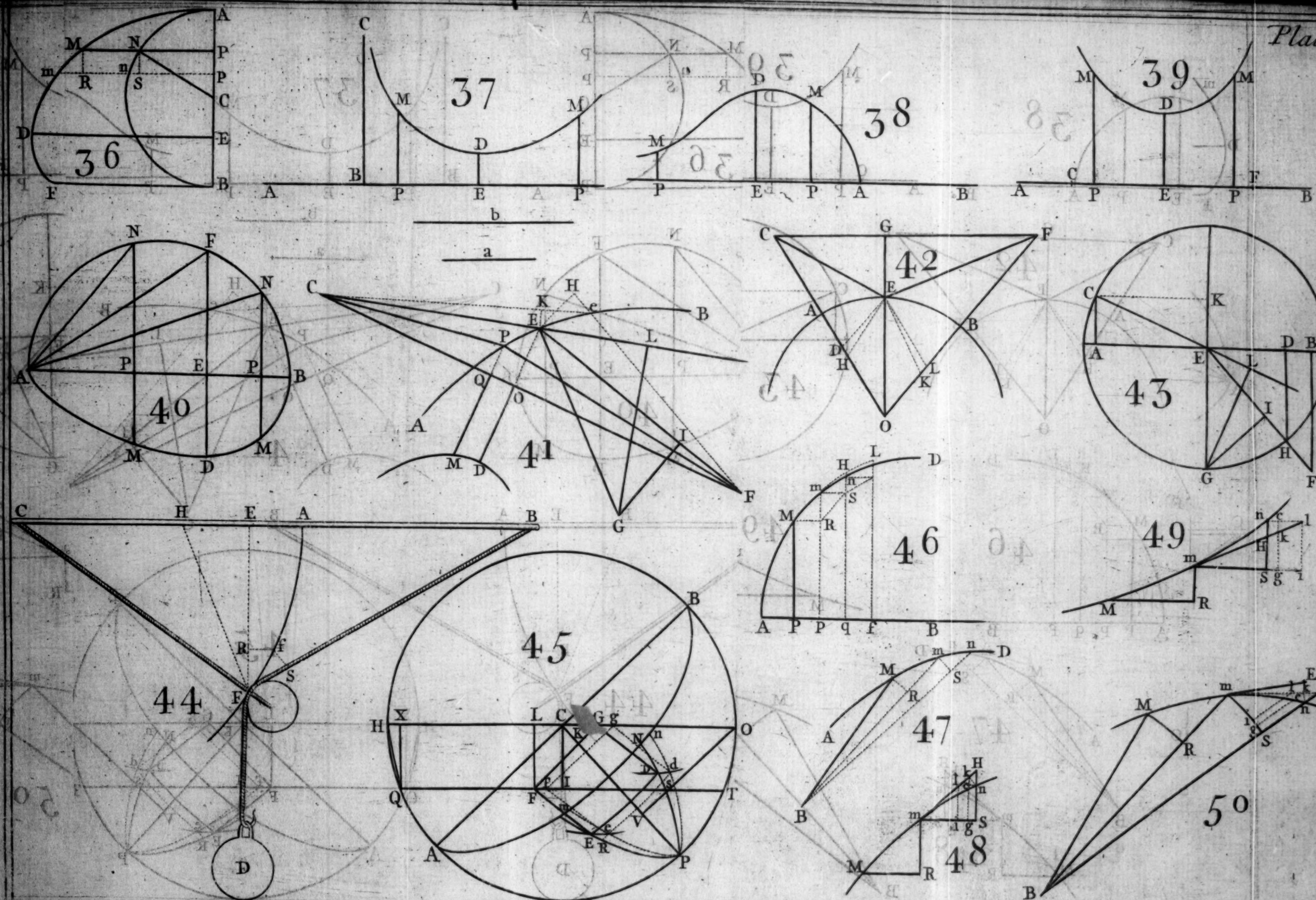
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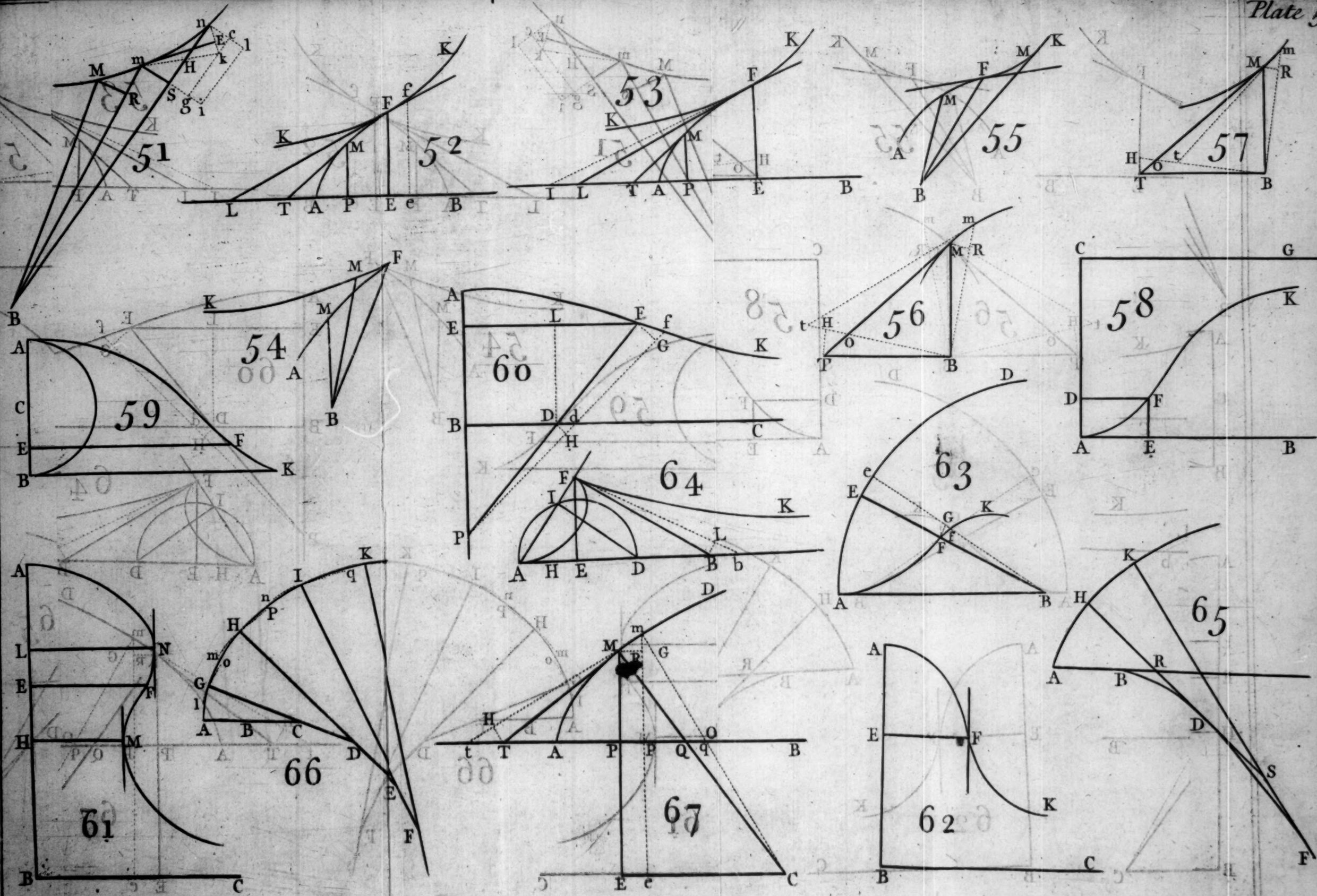


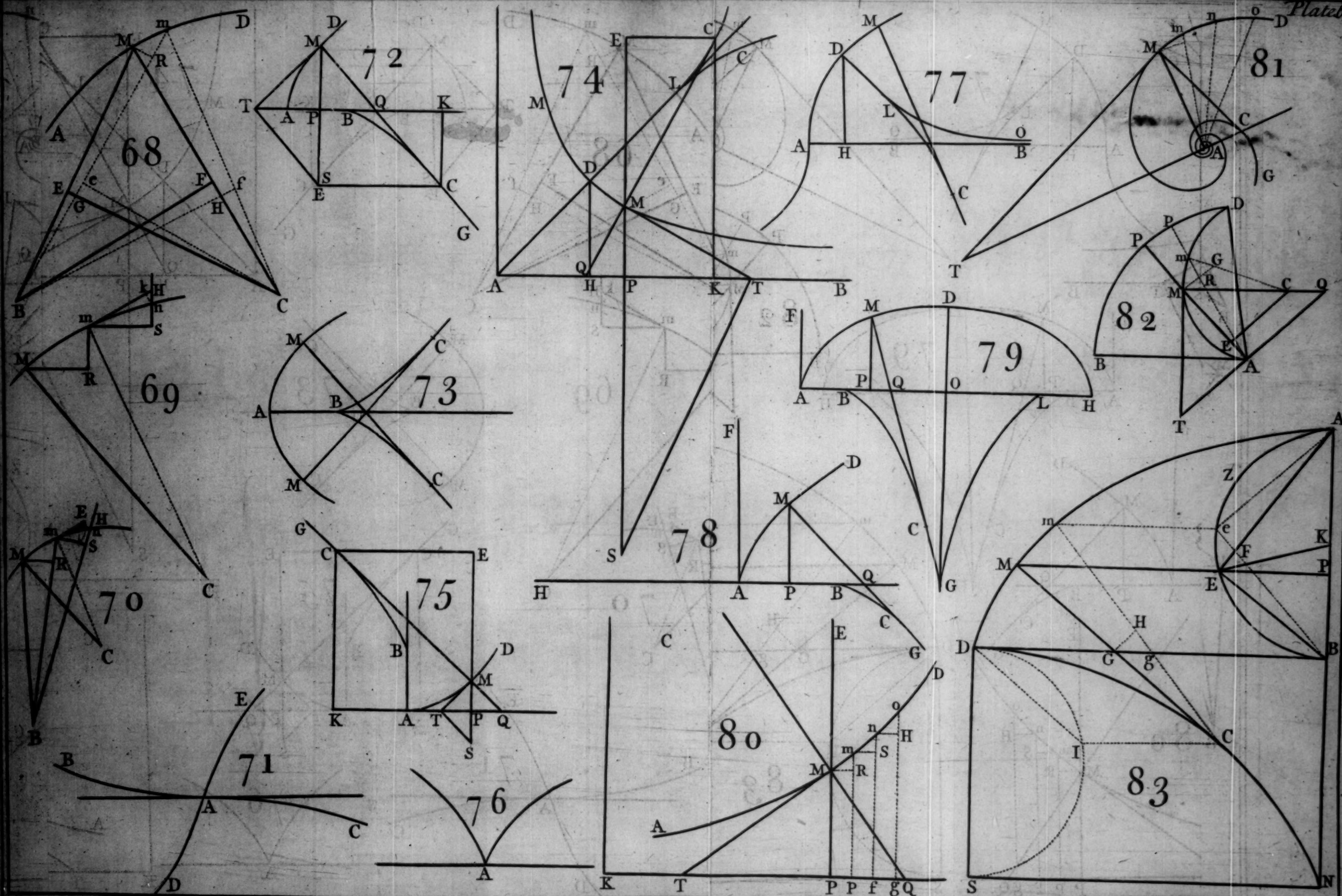


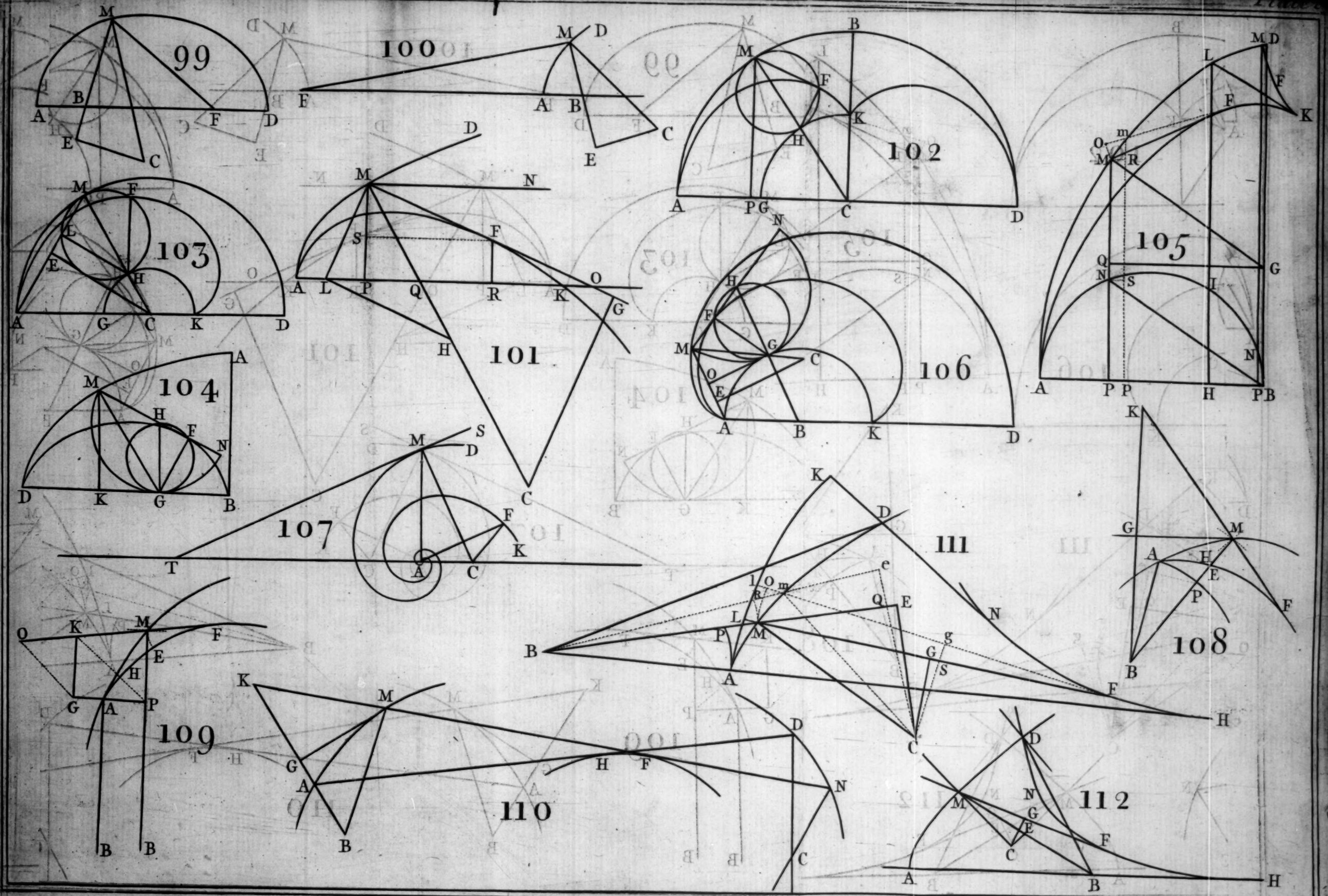


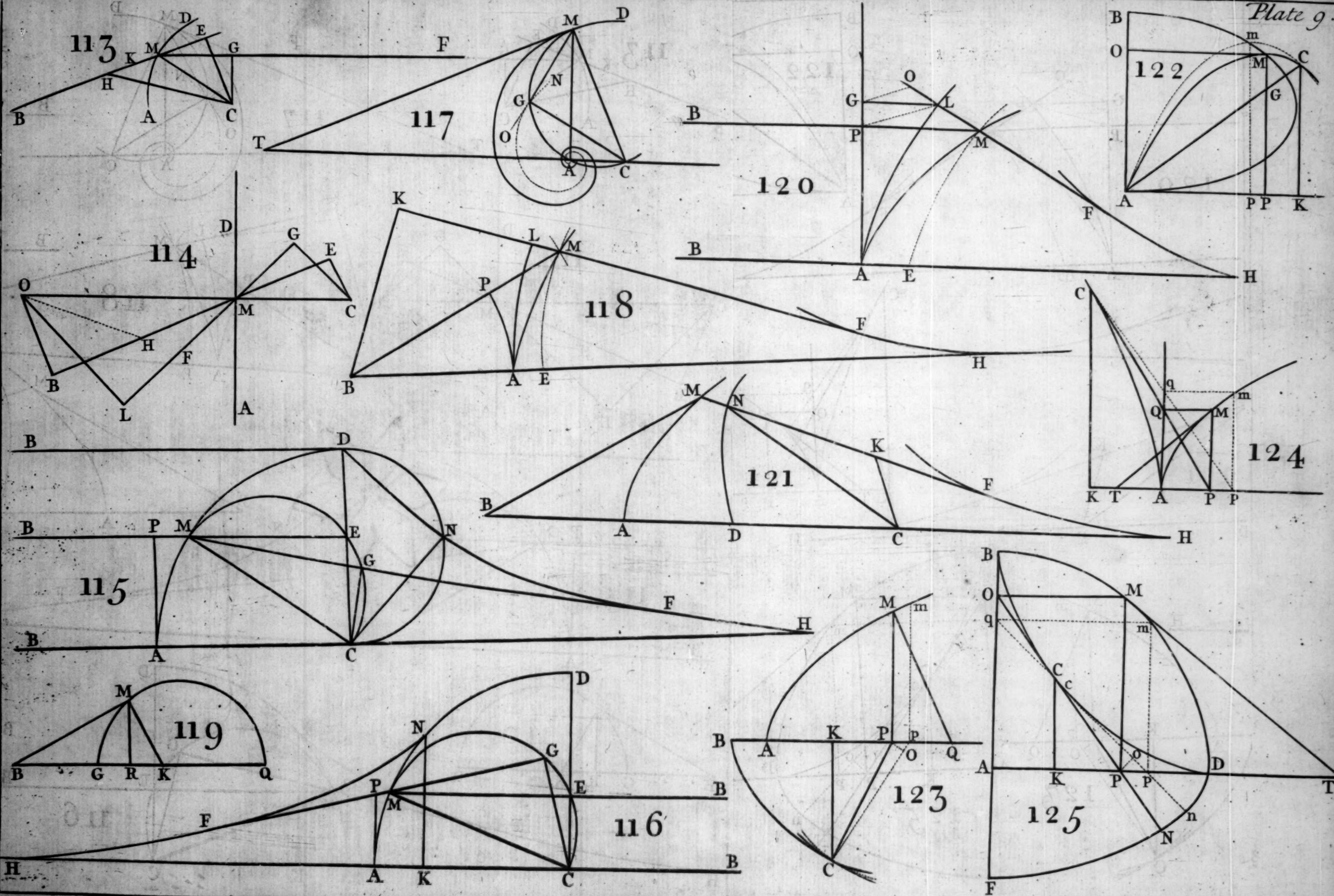


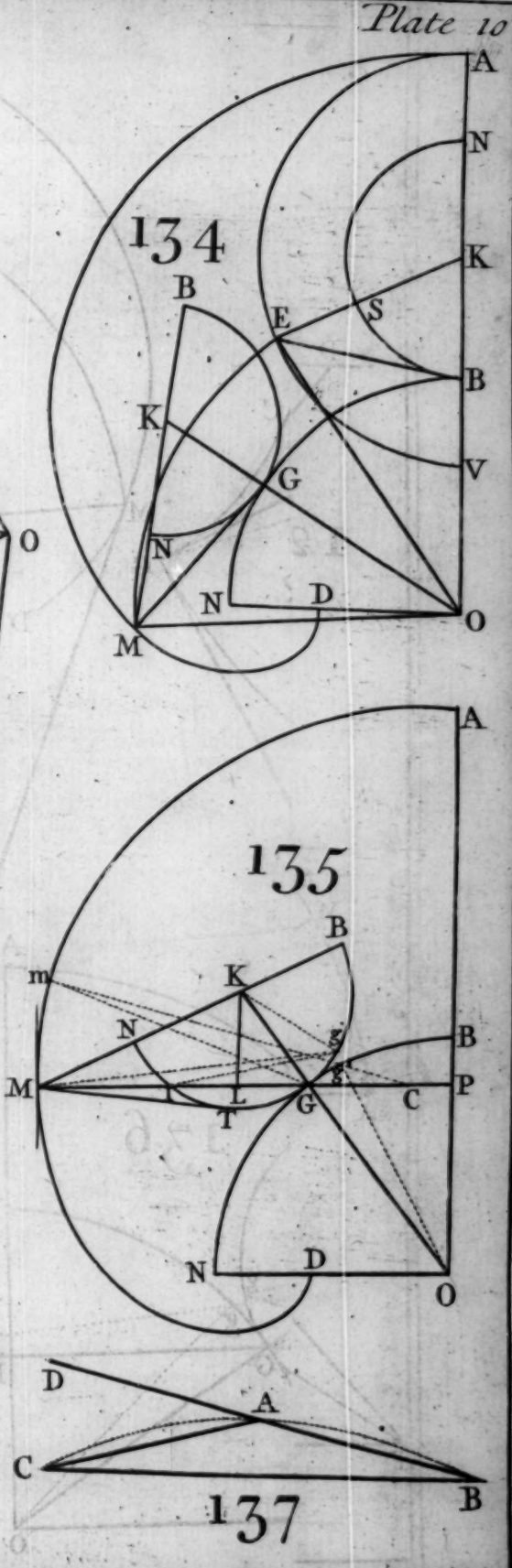
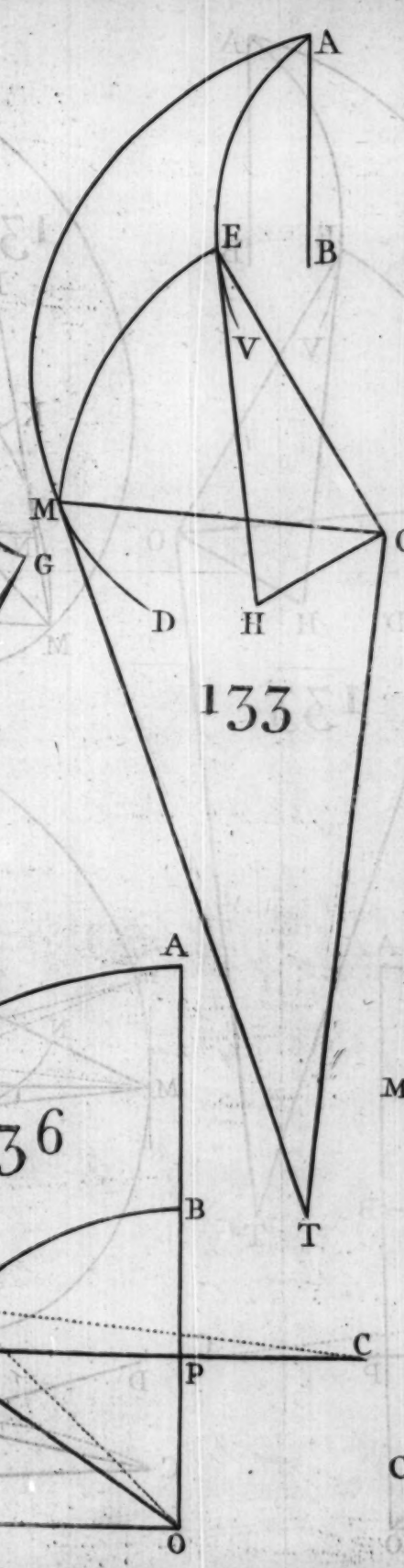
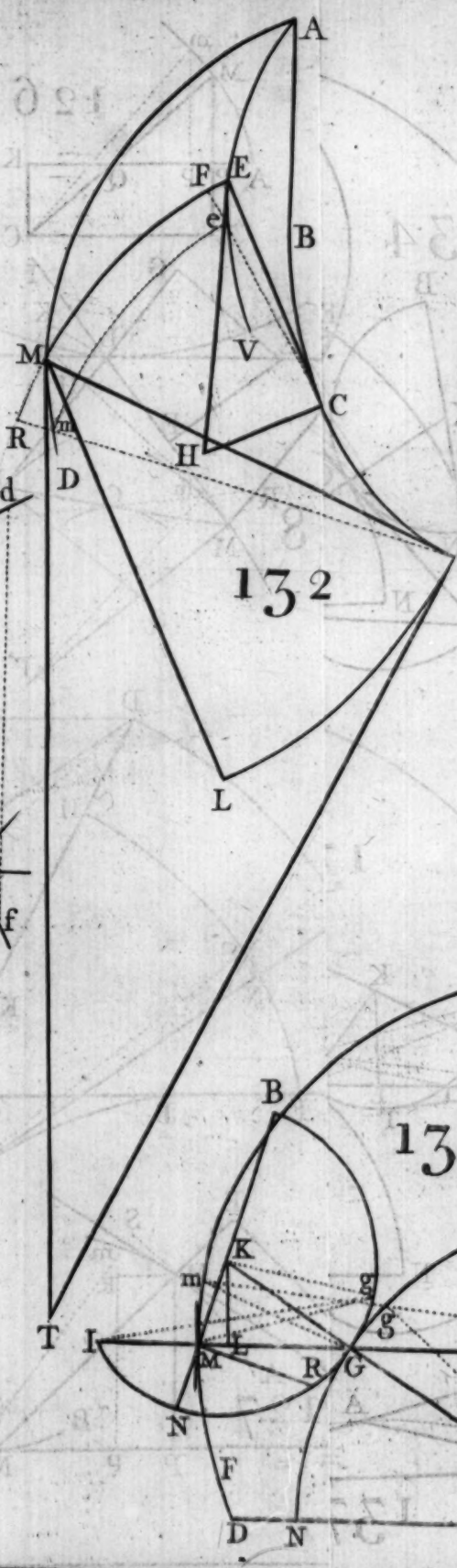
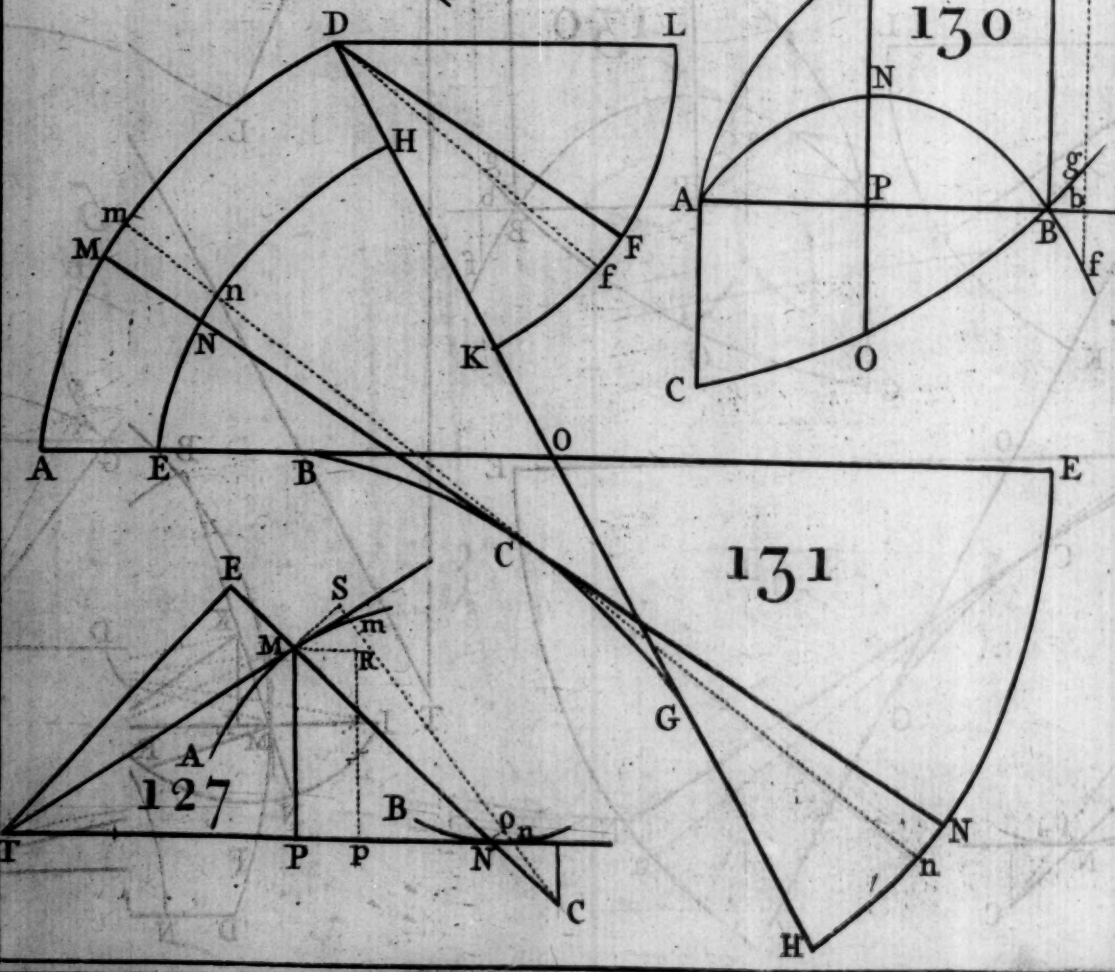
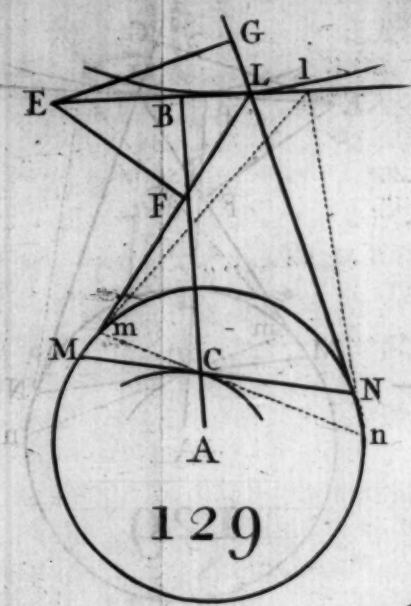
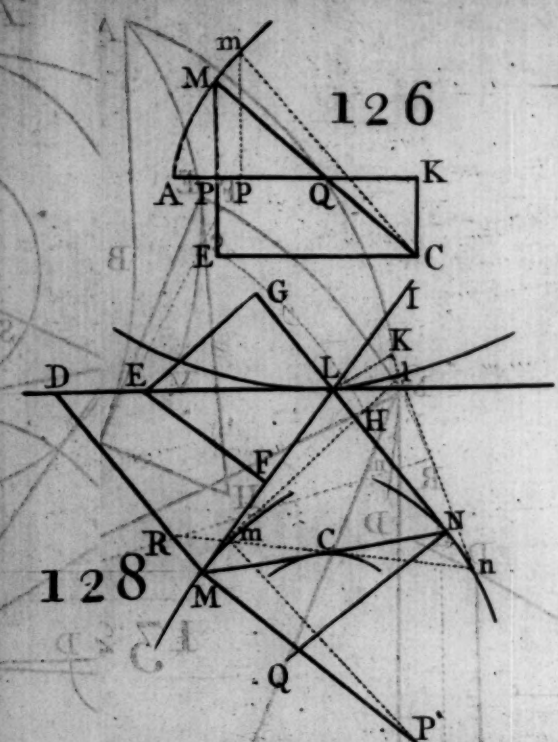


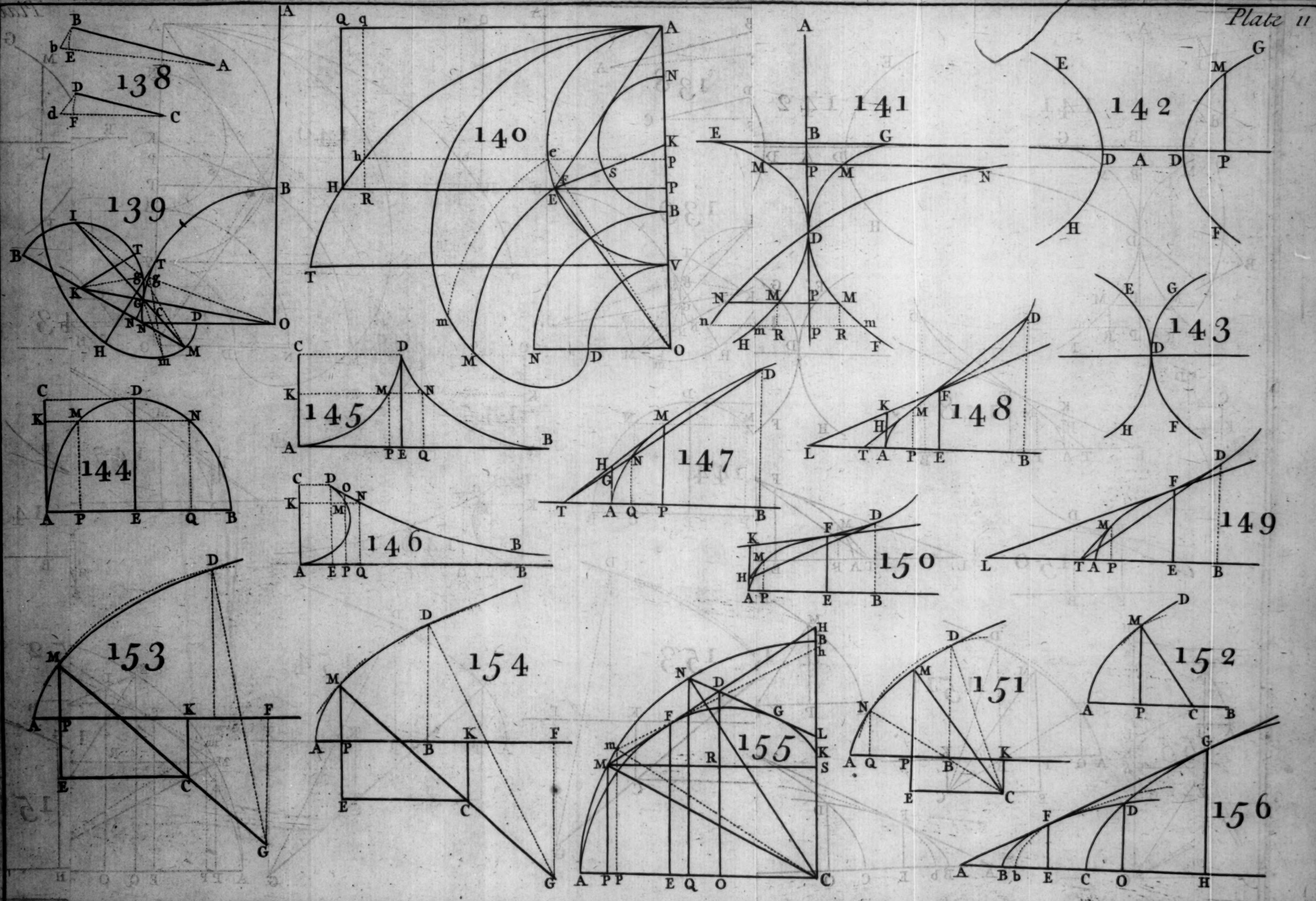


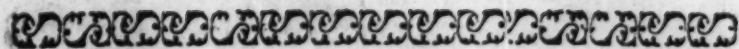












PART II.

Being an

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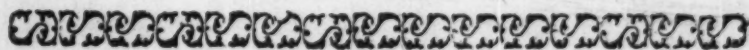
Containing the

Inverse Method of FLUXIONS,

WITH THE

Application thereof in the Investigation
of the Areas of Superficies, Lengths of
Curve Lines, Contents of Solids, and the
Determination of their Centres of Gravity
and Percussion.

Wherein are Examples of Solutions, accord-
ing to the excellent compendious Way of
the late Learned Mr. ROGER COTES, by
the Measures of Ratios and Angles, or Ta-
bles of Logarithms, and natural Sines and
Tangents.



EXAMINATION OF THE

PART II

APPENDIX

Inventor: Michael C. ...

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EXAMINATION OF THE



APPENDIX.

OF THE

Inverse Method of FLUXIONS.

SECTION I.

Of the Reduction of Fractional Expressions and Surd Quantities to infinite Series.



THE Fluents or flowing Quantities of Fluxions express'd fractionwise, or of such wherein there are Surds or radical Quantities, in general cannot be found till the said fractional Expressions are freed from their compound Denominator, and brought to simple ones, and the radical Quantities from their Surds, by throwing such Expressions into infinite Series. Which may be done by the two following Problems.

R

PROB.

PROB. I.

1. **T**O throw $\frac{b}{a+x}$ (*a and b being standing, and x a variable Quantity*) into an infinite Series, whereby it will be freed from its binomial Denominator.

Divide the Numerator *b* by the Denominator $a+x$, after the very same manner as you do decimal Fractions, by adding 0 to the Remainder, and repeating the Operation till you have gotten 4, 5 or 6 Terms in the Quotient; after which, in many Cases, you may find as many Terms as you please, by considering the Law of the Progression of those Terms already found. And an infinite Number or Series of Terms so found, will be the exact Quotient of the Division; but usually a few of the first Terms are sufficiently near the Truth for any Purpose.

EXAMPLE I.

$$a+x) b + 0 \left(\frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4}, \&c.$$

$$\frac{b + \frac{bx}{a}}{a}$$

$$0 - \frac{bx}{a} + 0$$

$$- \frac{bx}{a} - \frac{bx^2}{a^2}$$

$$0 + \frac{bx^2}{a^2} + 0$$

$$+ \frac{bx^2}{a^2} + \frac{bx^3}{a^3}$$

$$0 - \frac{bx^3}{a^3} + 0$$

$$- \frac{bx^3}{a^4} - \frac{bx^4}{a^4}$$

$$0 + \frac{bx^4}{a^4}, \&c.$$

For

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For dividing b by a , the Quotient is $\frac{b}{a}$. The Product of $\frac{b}{a}$ by $a + x$ is $\frac{ba}{a} + \frac{bx}{a}$ or $b + \frac{bx}{a}$. Which subtracted from the Dividend b , and there remains $0 - \frac{bx}{a}$. Again, if $0 - \frac{bx}{a}$ be divided by a , the Quotient will be $-\frac{bx}{a^2}$. Therefore the Product of $a + x$ into $-\frac{bx}{a^2}$, that is, $-\frac{abx}{a^2} - \frac{bx^2}{a^2}$, or $-\frac{bx}{a} - \frac{bx^2}{a^2}$ subtracted from the Dividend $-\frac{bx}{a}$ leaves $0 + \frac{bx^2}{a^2}$. Whence the Law of the Continuance of the Division is evident. Now the Quotient consists of an infinite Series of Terms, whose Numerators are the Powers of x , less by 1 than the Number of the Order multiply'd by b , and denominates the Powers of a , whose Exponents are equal to the Number of the Order of the Terms. For Example: In the third Term, the Exponent of the Power of x in the Numerator is 2, and of a in the Denominator is 3.

In like manner, if you put x the first Letter in the Divisor, and then divide b by $x + a$, as before, the Quotient will be $\frac{b}{x} - \frac{ba}{x^2} + \frac{ba^2}{x^3}$

$-\frac{ba^3}{x^4}$, &c. So that it's plain there will be as many Quotients or infinite Series gotten by this Division, as there are Terms in the Divisor; and those Terms of the Divisor which are greatest must stand first, as well as those in the Dividend, in order to have a true Series.

R 2

For

APPENDIX.

For Example: Let $b=1$, $x=1$, and $a=2$. Then if the Division be perform'd with a as the first Letter of the Divisor, you will find

$$\frac{1}{3} = \frac{1}{2+1} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16}, \text{ \&c. which}$$

we know to be true from other Principles. But if x be the first Letter, then will

$$\frac{1}{3} = \frac{1}{1+2} \text{ be } = 1 - 2 + 4 - 8 + 16, \text{ \&c.}$$

which is false. For this Series diverges, and differs so much the more from the true Quotient, as the Number of Terms is greater. For Example: One Term 1 exceeds $\frac{1}{3}$ by $\frac{2}{3}$; two Terms are deficient by $\frac{4}{3}$; three Terms exceeds it by $\frac{8}{3}$; four Terms are deficient by $\frac{16}{3}$, and so on.

EXAMPLE II.

$$b + x(aa + 0) \frac{aa}{b} - \frac{aax}{b^2} + \frac{aax^2}{b^3} - \frac{aax^3}{b^4}, \text{ \&c.}$$

$$aa + \frac{aax}{b}$$

$$0 - \frac{aax}{b} + 0$$

$$- \frac{aax}{b} - \frac{aax^2}{b^2}$$

$$0 + \frac{aax^2}{b^2} + 0$$

$$+ \frac{aa^2}{b^2} + \frac{aax^3}{b^3}$$

$$0 - \frac{aax^3}{b^3} + 0$$

$$- \frac{aax^3}{b^3} - \frac{aax^4}{b^4}$$

$$0 + \frac{aax^4}{b^4}, \text{ \&c.}$$

If

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If b be put first, then will the Quotient be
 $\frac{aa}{x} - \frac{aab}{x^2} + \frac{aab^2}{x^3} - \frac{aab^3}{x^4}, \&c.$

Again; the Quotient of $\frac{1}{1+xx}$ will be
 found $1 - x^2 + x^4 - x^6 + x^8, \&c.$ or (ma-
 king xx the first Term in the Divisor)
 $x^{-2} - x^{-4} + x^{-6} - x^{-8}, \&c.$

Moreover, $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x}$ freed from its com-
 pound Denominator by Division will be brought
 to $2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}}, \&c.$

And lastly, This Fraction
 $\frac{1 + \frac{1}{2}ax^2 - \frac{1}{8}a^2x^4 + \frac{1}{16}a^3x^6 - \frac{1}{128}a^4x^8, \&c.}{1 - \frac{1}{2}bx^2 - \frac{1}{8}b^2x^4 - \frac{1}{16}b^3x^6 - \frac{1}{128}b^4x^8, \&c.}$
 having both Numerator and Denominator in-
 finite Series may be freed from its compound
 Denominator, or brought into an infinite Se-
 ries by dividing the Numerator by the Deno-
 minator, as before: the Operation being alike
 to that whereby one interminate decimal Fra-
 ction is divided by another.

This Quotient or infinite Series will be
 $1 + \frac{1}{2}bx^2 + \frac{3}{8}b^2x^4 + \frac{5}{16}b^3x^6 + \frac{35}{128}b^4x^8, \&c.$

$$\begin{aligned} &+ \frac{1}{2}a - \frac{1}{4}ab^2 + \frac{3}{16}ab^2 + \frac{5}{32}ab^3 \\ &- \frac{1}{8}a^2 - \frac{1}{16}a^2b - \frac{3}{64}a^2b^2 \\ &+ \frac{1}{16}a^3 + \frac{1}{32}a^3b \\ &- \frac{1}{128}a^4. \end{aligned}$$

PROB. II.

2. **T**O free a compound Expression from Surds,
 by throwing it into an infinite Series. Sup-
 pose $\sqrt{aa+xx}$.

Extract

Extract the Root thereof, as you do the Root of a decimal Fraction by the Addition of Cyphers, and bringing out so many Terms, till you may discover the Law of the Progression; that from those already found you may continue on the Terms at pleasure, and the thing is done.

EXAMPLE I.

$$aa + xx \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}, \&c. \right)$$

$$\begin{array}{r}
 aa \\
 \hline
 0 + xx \\
 \quad xx + \frac{x^3}{4a^2} \\
 \hline
 0 - \frac{x^4}{4a^2} \\
 \quad - \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6} \\
 \hline
 0 + \frac{x^6}{8a^4} - \frac{x^8}{64a^6} \\
 \quad \frac{x^6}{8a^4} + \frac{x^8}{16a^6} - \frac{x^{10}}{64a^8} + \frac{x^{12}}{256a^{10}} \\
 \hline
 0 - \frac{5x^8}{64a^6} + \frac{x^{10}}{64a^8} - \frac{x^{12}}{256a^{10}}, \&c.
 \end{array}$$

For the square Root of aa is $+a$, for the first Letter or Term of the Root. Which squar'd and subtracted from $aa + xx$, there remains xx . Which being divided by $2a$ (as in the Extraction of the square Root) and the Quotient will be $+\frac{x^2}{2a}$ the second Term of the Root. Which added to $2a$, and the whole multiplied by $\frac{x^2}{2a}$, will be $xx + \frac{x^4}{4a^2}$. This ta-

ken

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ken from xx , and there remains $-\frac{x^4}{4a^2}$. Which divided by $2a - \frac{x^2}{a}$ the Double of the two first Terms, and the Quotient $-\frac{x^4}{8a^3}$ will be the third Term of the Root. This added to $2a - \frac{x^2}{a}$, and the whole multiplied by $-\frac{x^4}{8a^3}$, will be $-\frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}$. Which taken from $-\frac{x^4}{4a^2}$, and there remains $0 + \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$.

And after this manner you may find any Number of Terms. And so continue the Series on at Pleasure.

If xx be the first Term of the Surd, then will the Root be $x + \frac{ax}{2x} - \frac{a^2}{8x^3} + \frac{a^6}{16x^5} - \frac{5a^8}{128x^7}$, &c.

And here, as well as in the Division foregoing, the greatest Term of the Surd must stand first, otherwise the Root or Series will not be true.

EXAMPLE II.

$$x - xx \left(x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{12}x^{\frac{7}{2}} - \frac{1}{128}x^{\frac{9}{2}}, \&c. \right)$$

$$\begin{array}{r} x \\ \hline 0 - x^2 \\ \hline -x^2 + \frac{1}{4}x^3 \\ \hline 0 - \frac{1}{4}x^3 \\ \hline -\frac{1}{4}x^3 + \frac{1}{8}x^4 + \frac{1}{24}x^5 \\ \hline 0 - \frac{1}{8}x^4 - \frac{1}{24}x^5 \\ \hline -\frac{1}{8}x^4 + \frac{1}{16}x^5 + \frac{7}{24}x^6 + \frac{1}{256}x^7 \\ \hline 0 - \frac{1}{24}x^5 - \frac{1}{64}x^6 - \frac{1}{256}x^7 \\ \hline -\frac{1}{24}x^5 + \frac{1}{128}x^6 + \frac{1}{312}x^7 + \frac{1}{1024}x^8 + \frac{1}{16384}x^9 \\ \hline 0 - \frac{1}{128}x^6 - \frac{1}{312}x^7 - \frac{1}{1024}x^8 - \frac{1}{16384}x^9 \\ \hline \&c. \end{array}$$

EXAMPLE III.

$$\begin{array}{r}
 1 - 2x + 3x^2 - 4x^3 + 5x^4, \&c. \quad (1 - x + x^2, \&c. \\
 \hline
 1 \\
 0 - 2x + 3x^2 - 4x^3 + 5x^4, \&c. \\
 \hline
 -2x + x^2 \\
 0 + 2x^2 - 4x^3 + 5x^4, \&c. \\
 \hline
 2x^2 - 2x^3 + x^4 \\
 0 - 2x^3 - 4x^4, \&c.
 \end{array}$$

After the same way you may extract the Cube, Biquadrate, &c. Root of a Surd, even if it be an infinite Series.

But these Extractions, as well as the Divisions aforegoing, will be very much shorten'd by a Theorem invented for that Purpose by

Sir Isaac Newton, which is this: $P + P Q^{\frac{m}{n}}$

$$\begin{aligned}
 &= P^{\frac{m}{n}} + \frac{m}{n} A Q + \frac{m-n}{2n} B Q^2 + \frac{m-2n}{4n} C Q^3 \\
 &+ \frac{m-3n}{4n} D Q^4 + \&c.
 \end{aligned}$$

Where $P + P Q$ is the Quantity whose Root or any Dimension, or Root of the Dimension, is to be found; that is, express'd by an infinite Series. P is the first Term of that Quantity: Q the rest of the Terms divided by first; and $\frac{m}{n}$ is the numeral Index of the Dimension of $P + P Q$. Moreover, $A, B, C, D, \&c.$ are used for the Terms found in the Quotient, viz. A for the first Term $P^{\frac{m}{n}}$; B for the second $\frac{m}{n} A Q$, and so on.

A few

APPENDIX.

9

A few Examples will shew this wonderful Theorem's Use.

EXAMPLE I.

$$\sqrt{aa + xx} \text{ or } aa + xx^{\frac{1}{2}} \text{ is } = a + \frac{xx}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}, \&c. \text{ for in this Case, } P \text{ is } = aa, \\ \mathcal{Q} = \frac{xx}{aa}, m = 1, n = 2, A (= P^{\frac{m}{n}} = aa^{\frac{1}{2}}) = a. \\ B = \frac{m}{n} A \mathcal{Q} = \frac{xx}{2a}. C (= \frac{m-n}{2n} B \mathcal{Q}) = -\frac{x^4}{8a^3}, \&c.$$

EXAMPLE II.

$$\sqrt[5]{a^5 + a^4x - x^5}, \text{ or } a^5 + a^4x - x^5^{\frac{1}{5}} \text{ is } = a + \frac{a^4x - x^5}{5a^4} - \frac{2a^8xx + 4a^4x^6 - 2x^{10}}{25a^9} + \&c. \text{ for here } m = 1, n = 5, P = a^5, \text{ and } \mathcal{Q} = \frac{a^4x - x^5}{a^5}.$$

EXAMPLE III.

$$\frac{b}{\sqrt{y^3 - a^2y}}, \text{ or } b \times y^3 - a^2y^{-\frac{1}{2}} \text{ is } = b \times \frac{1}{y} + \frac{aa}{3y^3} + \frac{2a^4}{9y^5} + \frac{14a^6}{81y^7}, \&c. \text{ for } P = y^3. \mathcal{Q} = -\frac{aa}{yy}. m = -1, n = 3. A (P^{\frac{m}{n}} = y^3 \times -\frac{1}{3}) = y^{-1}, \\ \text{that is, } \frac{1}{y}. B (= \frac{m}{n} A \mathcal{Q} = -\frac{1}{3} \times \frac{1}{y} \times -\frac{aa}{yy}) = \frac{aa}{3y^3}, \&c.$$

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EXAMPLE IV.

$$\sqrt[3]{a+x^4} \text{ or } a+x^{\frac{4}{3}} \text{ is } = a^{\frac{1}{3}} + \frac{4x a^{\frac{1}{3}}}{3} + \frac{2xx}{9a^{\frac{2}{3}}} - \frac{4x^3}{81a^{\frac{5}{3}}} + \&c. \text{ for } P=a. \quad Q=\frac{x}{a}. \quad m=4, \\ n=3. \quad A\left(=P^{\frac{m}{n}}\right)=a^{\frac{4}{3}}, \&c.$$

EXAMPLE V.

$$a+x^5 \text{ or } a+x^{\frac{5}{1}} \text{ is } = a^{\frac{1}{1}} + 5a^{\frac{4}{1}}x + 10a^{\frac{3}{1}}xx + 10a^{\frac{2}{1}}x^3 + 5a^{\frac{1}{1}}x^4 + x^5. \text{ For } P=a. \quad Q=\frac{x}{a}. \\ m=5, \text{ and } n=1. \quad A\left(=P^{\frac{m}{n}}\right)a^{\frac{1}{1}}. \quad B\left(=\frac{m}{n}A Q\right)=5a^{\frac{4}{1}}x. \text{ and so } C=10a^{\frac{3}{1}}xx. \quad D=10a^{\frac{2}{1}}ax^3. \quad E=5a^{\frac{1}{1}}x^4. \quad F=x^5. \text{ and } G\left(=\frac{m-5n}{6n}FQ\right)=0.$$

EXAMPLE VI.

$$\frac{1}{a+x} \text{ or } a+x^{-1} \text{ or } a+x^{-\frac{1}{1}} \text{ is } = \frac{1}{a} - \frac{x}{aa} + \frac{xx}{a^3} - \frac{x^3}{a^4} + \&c. \text{ for in this Case } P=a, \\ Q=\frac{x}{a}, m=-1, n=1. \text{ and } A\left(=P^{\frac{m}{n}}\right)=a^{-1} \\ a^{-\frac{1}{1}}\right)=a^{-1} \text{ or } \frac{1}{a}. \quad B\left(=\frac{m}{n}A Q\right)=-1 \\ \times \frac{1}{a} \times \frac{x}{a}\right)=-\frac{x}{aa} \text{ and } C=\frac{xx}{a^3}. \quad D=-$$

APPENDIX.

II

$\frac{x^3}{a^4}$, &c. So that the said wonderful Theorem likewise frees Fractions from their Denominators, as well as extracts Roots.

EXAMPLE VII.

$$\frac{3}{a+x} = 3 \times \frac{1}{a+x} \text{ is } = \frac{1}{a^{\frac{1}{3}}} - \frac{3x}{a^{\frac{4}{3}}} + \frac{6xx}{a^{\frac{7}{3}}} - \frac{10x^3}{a^{\frac{10}{3}}}, \&c.$$

EXAMPLE VIII.

$$\text{And } \frac{b}{\sqrt[3]{a+x}} = b \times \frac{1}{\sqrt[3]{a+x}} \text{ is } = b \times \frac{1}{a^{\frac{1}{3}}} - \frac{x}{3a^{\frac{4}{3}}} + \frac{2xx}{9a^{\frac{7}{3}}} - \frac{14x^3}{81a^{\frac{10}{3}}}, \&c.$$

EXAMPLE IX.

$$\text{And } \frac{b}{\sqrt[5]{a+x}} = b \times \frac{1}{\sqrt[5]{a+x}} \text{ is } = b \times \frac{1}{a^{\frac{1}{5}}} - \frac{3x}{5a^{\frac{6}{5}}} + \frac{12xx}{25a^{\frac{11}{5}}} - \frac{52x^3}{125a^{\frac{16}{5}}}, \&c.$$

In the Philosophical Transactions, N°. 230. Mr. *De Moivre* has given us the following Theorem for raising an infinite Series to a given Power m , or extracting the Root thereof, viz.

$$\begin{aligned} \frac{az + bz^2 + cz^3 + dz^4 + ez^5 + fz^6, \&c.}{a^m} \text{ will be } &= \\ &= a^m z^m + \\ &+ \frac{m}{1} a^{m-1} b z^{m+1} \\ &+ \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 z^{m+2} \\ &+ \frac{m}{1} a^{m-1} c z^{m+3} \end{aligned} \left. \vphantom{\begin{aligned} &+ \frac{m}{1} a^{m-1} b z^{m+1} \\ &+ \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 z^{m+2} \\ &+ \frac{m}{1} a^{m-1} c z^{m+3} \end{aligned}} \right\} z^{m+3}$$

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$$\begin{aligned}
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 \\
 & + \frac{m}{1} \times \frac{m-1}{1} a^{m-2} bc \\
 & + \frac{m}{1} a^{m-1} d
 \end{aligned}
 \left. \vphantom{\begin{aligned} & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 \\ & + \frac{m}{1} \times \frac{m-1}{1} a^{m-2} bc \\ & + \frac{m}{1} a^{m-1} d \end{aligned}} \right\} z^{m+1}$$

$$\begin{aligned}
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 \\
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{1} a^{m-3} b^2 c \\
 & + \frac{m}{1} \times \frac{m-1}{1} a^{m-2} b d \\
 & + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} c^2 \\
 & + \frac{m}{1} a^{m-1} e
 \end{aligned}
 \left. \vphantom{\begin{aligned} & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 \\ & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{1} a^{m-3} b^2 c \\ & + \frac{m}{1} \times \frac{m-1}{1} a^{m-2} b d \\ & + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} c^2 \\ & + \frac{m}{1} a^{m-1} e \end{aligned}} \right\} z^{m+4}$$

$$\begin{aligned}
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5} b^5 \\
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{1} a^{m-4} b^3 c \\
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{1} a^{m-3} b^2 d \\
 & + \frac{m}{1} \times \frac{m-1}{1} \times \frac{m-2}{2} a^{m-3} b c^2 \\
 & + \frac{m}{1} \times \frac{m-1}{1} a^{m-2} b e \\
 & + \frac{m}{1} \times \frac{m-1}{1} a^{m-2} c d \\
 & + \frac{m}{1} a^{m-1} f
 \end{aligned}
 \left. \vphantom{\begin{aligned} & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5} b^5 \\ & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{1} a^{m-4} b^3 c \\ & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{1} a^{m-3} b^2 d \\ & + \frac{m}{1} \times \frac{m-1}{1} \times \frac{m-2}{2} a^{m-3} b c^2 \\ & + \frac{m}{1} \times \frac{m-1}{1} a^{m-2} b e \\ & + \frac{m}{1} \times \frac{m-1}{1} a^{m-2} c d \\ & + \frac{m}{1} a^{m-1} f \end{aligned}} \right\} z^{m+5}$$

$$\begin{aligned}
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times \frac{m-5}{6} a^{m-6} b^6 \\
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{1} a^{m-5} b^4 c \\
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{1} a^{m-4} b^3 d \\
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{1} \times \frac{m-3}{2} a^{m-4} b^2 c^2 \\
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{1} a^{m-3} b^2 e \\
 & + \frac{m}{1} \times \frac{m-1}{1} \times \frac{m-2}{1} a^{m-3} b c d \quad \left. \begin{array}{l} z^{m+6}, \\ \&c. \end{array} \right\} \\
 & + \frac{m}{1} \times \frac{m-1}{1} a^{m-3} b f \\
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} c^3 \\
 & + \frac{m}{1} \times \frac{m-1}{1} a^{m-2} c e \\
 & + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} d^2 \\
 & + \frac{m}{1} a^{m-1} g
 \end{aligned}$$

For understanding of which, it is only necessary to consider all the Terms by which the same Power of z is multiply'd: In order to which two things in each of these Terms must be consider'd, 1^o, The Product of certain Powers of the given Quantities or Coefficients a, b, c, d , &c. And, 2^o, The *Unciæ* or Products of $\frac{m}{1} \times \frac{m-1}{2}$, &c. prefixed to them.

Now to find all the Products belonging to the same Power of z . For Example; to find that

that Product whose Index is $m+r$ (r being any whole Number) the said Products must be distinguished into several Classes. Those which immediately after some certain Power of a (by which all these Products begin) are Products of the first Class: as $a^{m-1}b^3e$ is a Product of the first Class, because b immediately follows a^{m-4} . Those which immediately after some Power of a have c , are Products of the second Class. So $a^{m-3}ccd$ is a Product of the second Class: Those which immediately after some Power of a have d , are Products of the third Class, and so of the rest.

This being understood, 1^o, Multiply all the Products belonging to z^{m+r-1} (which immediately precedes z^{m+r}) by b , and divide them all by a . 2^o, Multiply by c , and divide by a , all the Products belonging to z^{m+r-2} except those of the first Class. 3^o, Multiply by d , and divide by a all the Products belonging to z^{m+r-3} except those of the first and second Class. 4^o, Multiply by e , and divide by a , all the Terms belonging to z^{m+r-4} , except those of the first, second, and third Class; and so on, till you meet twice with the same Term. Lastly, Add the Product of a^{m-1} into the Letter whose Exponent is $r+1$ to all these Terms.

Note, The Exponent of a Letter is the Number expressing what Place that Letter has in the Alphabet, as 3 is the Exponent of the Letter c .

By this Rule it is manifest that it is easy to find all the Products belonging to the several Powers of z , if you have but the Product belonging to z^m , viz. a^m .

Now

Now to find the *Uncie* prefix'd to every Product, you must consider the Sum of the Units contained in the Exponents of the Letters that compose it (the Index of *a* excepted); then I write as many Terms of the Series $m \times m - 1 \times m - 2 \times m - 3$, &c. as there are Units in the Sum of these Indexes; this Series is to be the Numerator of a Fraction, whose Denominator is the Product of the several Series $1 \times 2 \times 3 \times 4 \times 5$, &c. $1 \times 2 \times 3 \times 4 \times 5$, &c. $1 \times 2 \times 3 \times 4 \times 5 \times 6$, &c. the first of which contains as many Terms as there are Units in the Index of *b*; the second as many as there are Units in the Index of *c*; the third as many as there are Units in the Index of *d*, &c.

The Demonstration of this see in the above cited Transaction.

Here follows an Example or two of the Use of this Theorem.

EXAMPLE. I.

To raise this infinite Series $\frac{1}{x} + \frac{1}{xx} + \frac{1}{x^3} + \frac{1}{x^4}$, &c. to the second Power, or to square it.

In this Case in the Theorem $m = 2$, $z = x$, $a = \frac{1}{xx}$, $b = \frac{1}{x^4}$, $c = \frac{1}{x^6}$, $d = \frac{1}{x^8}$, &c. therefore

$\frac{1}{x} + \frac{1}{xx} + \frac{1}{x^3} + \frac{1}{x^4}$, &c. will be $\frac{1}{xx} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{4}{x^5}$, &c. for the first Term ($a^m z^m (= \frac{1}{x^4} \times xx)$)

is $= \frac{1}{xx}$. The second Term $\frac{m}{1} a^{m-1} b z^{m-1}$

($= \frac{2}{xx} \times \frac{1}{x^4} \times x^3$) is $= \frac{2}{x^3}$.

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The Third $\left. \begin{aligned} &\frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 \\ &+ \frac{m}{1} a^{m-1} c \end{aligned} \right\} z^{m+2} \left\{ = 2 \times \frac{1}{2} \times \frac{1}{x^6} \times x^4 \right.$

$= \frac{1}{x^4} + 2 \times \frac{1}{xx} \times \frac{1}{x^6} \times x^4 = \frac{2}{x^4} \left. \right\} \text{is} = \frac{3}{x^4}.$

The Fourth Term $\left. \begin{aligned} &\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 \\ &\frac{m}{1} + \frac{m-1}{1} a^{m-2} b c \\ &\frac{m}{1} a^{m-1} d \end{aligned} \right\} z^{m+3} = \left\{ \right.$

$2 \times \frac{1}{2} \times \frac{0}{3} \times xx \times \frac{1}{x^{12}} \times x^5 = 0$

$2 \times 1 \times 1 \times \frac{1}{x^4} \times \frac{1}{x^6} \times x^5 = \frac{2}{x^5}$

$2 \times \frac{1}{xx} \times \frac{1}{x^8} \times x^5 = \frac{2}{x^5} \left. \right\} \text{is} = \frac{4}{x^5}.$

EXAMPLE II.

To square this infinite Series $1 - x + x^2 + x^3 + x^4, \&c.$

In this Case in the Theorem $m=2, z=x.$

$a = \frac{1}{x} - 1. b = 1. c = -1. d = 1, \&c.$ and

so $1 - x + x^2 - x^3 + x^4, \&c.$ will be $= 1 - 2x + 3x^2 - 4x^3 + 5x^4, \&c.$ for $a^m z^m$

$\left(= \frac{1}{x} - 1 \times x^2 \right) = 1 - 2x + xx. \frac{m}{1} a^{m-1}$

$b z^{m+1} \left(= \frac{2}{1} \times \frac{1}{x} - x x^3 \right) = 2xx - 2x^3.$

$\frac{m}{1}$

$$\left. \begin{array}{l} \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 \\ \frac{m}{1} a^{m-1} c \end{array} \right\} z^{m+2} = \left\{ \begin{array}{l} \frac{2}{1} \times \frac{1}{2} \times \frac{1}{x} - 1 \times 1 \times x^4 \\ \frac{2}{1} \times \frac{1}{x} - 1 \times -1 \times x^4 \end{array} \right\} = -x^3 + 2x^4, \&c.$$

EXAMPLE III.

To raise $1 - x + x^3 - x^5 + x^7, \&c.$ to the third Power, or to cube it.

Here $m = 3$. $z = x$. $a = \frac{1}{x} - 1$, $b = 0$, $c = 1$, $d = 0$. and so the third Power will be $1 - 3x + 3x^2 - 6x^4, \&c.$ for $a^m z^m$
 $\left(= \frac{1}{x} - 1 \times x^3 \right) = 1 - 3x + 3x^2 - x^3.$

$$\frac{m}{1} a^{m-1} b z^{m+1} \left(= 3 \times \frac{1}{x} - 1 \times 0 \times x^4 \right) = 0.$$

$$\left. \begin{array}{l} \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 \\ \frac{m}{1} a^{m-1} c \end{array} \right\} z^{m+2} = \left\{ \begin{array}{l} 3 \times 1 \times \frac{1}{x} - 1 \times 0^2 \times x^5 = 0 \\ 3 \times \frac{1}{x} - 1 \times 1 \times x^5 = 3x^3 - 6x^4 \times 3x^5 \end{array} \right\} = 3x^3 - 6x^4 + 3x^5.$$

And because $b = 0$, and also d , therefore the next Term of the general Theorem will be 0. And thus you may proceed on.

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EXAMPLE IV.

To extract the Root of an infinite Series; that is, if z be $= ax + bx^2 + cx^3 + dx^4 + ex^5$, &c. to find the Value of x in an infinite Series of Terms affected with z , and free from x .

First, Let us suppose $x = fz + bz^2 + kz^3 + lz^4 + mz^5 + nz^6$, &c. Then by the Theorem $x^2 = f^2 z^2 + 2fbz^3 + b^2 z^4 + 2bkz^5 + k^2 z^6 + 2fklz^6 + 2fmz^6$ &c.

$$x^3 = f^3 z^3 + 3f^2 bz^4 + 3fb^2 z^5 + b^3 z^6, \text{ \&c.}$$

$$+ 3f^2 kz^5 + 3f^2 lz^6 + 6fbkz^6$$

$$x^4 = f^4 z^4 + 4f^3 bz^5 + 6f^2 b^2 z^6, \text{ \&c.}$$

$$x^5 = f^5 z^5 + 5f^4 bz^6$$

$$x^6 = f^6 z^6, \text{ \&c.}$$

Now substitute these Values in the Equation $0 = -z + ax + bx^2 + cx^3 + dx^4 + ex^5$, &c. and then will $-z = -z$.

$$+ax = +afz + abz^2 + akz^3 + alz^4 + amz^5 + anz^6, \text{ \&c.}$$

$$+bx^2 = *bf^2 z^2 + 2bfbz^3 + bb^2 z^4 + 2bflz^5 + bk^2 z^6, \\ + 2bbkz^4 +$$

$$+ 2bbkz^5 + 2bfmz^6$$

$$+cx^3 = * * +cf^3 z^3 + 3cf^2 bz^4 + cf^2 z^5 + 3cf^2 lz^6 + 3cf^2 kz^5 + \text{ \&c.}$$

$$3cf^2 lz^6 + 6fbkz^6$$

$$dx^4 = * * * +dfz^4 + 4df^3 z^5 + 6df^2 b^2 z^6 + 4df^3 kz^6$$

$$ex^5 = * * * * +ef^5 z^5 + 5ef^4 bz^6$$

Now if the Sum of the Coefficients of every Term in this Equation be made equal to no-

nothing, we may get the Values of the Coefficients f, b, k, l, m, n , thus. The Sum of the

Coefficients of the first Term $\frac{-1}{af} \} z$ will be $af - 1$. Whence if $af - 1 = 0$, then will f be $= \frac{1}{a}$. In like manner the Sum of the

Coefficients of the second Term $\frac{ab}{bf^2} \} z^2$ will be $ab + bf^2$. Whence if $ab + bf^2 = 0$, there will arise $b = \frac{-bf^2}{a} = \frac{-b}{a^3}$. In like manner

the Sum of the Coefficients of the third Term made equal to 0 will be $ak + 2bfb + cf^3 = 0$.

Whence $k = \frac{-2bfb - cf^3}{a} = \frac{2b^2 - ac}{a^5}$. Again,

$al + bb + 2bfb + 3cf^2b + df^4 = 0$. Whence $l =$

$$\frac{-bb - 2bfb + 3cfb - df^4}{a} = \frac{-b^3}{a^7 - 4b^3} + \frac{a^7 + 2ba}{a^6 + 3bc}$$

$$+ \frac{a^6 - d}{a^5} = \frac{5abc - 5b^3 - a^2d}{a^7}$$

$$\text{So likewise } m \text{ is } = \frac{14b^4 + 6abd - 21ab^2c + 3a^2c^2 - a^3e}{a^9} \text{ And } n =$$

$$\frac{-42b^5 + 84ab^3c - 28a^2bc - 28a^2bd + 7a^3cd + 7a^3cd + 7a^3bea^4f}{a^{11}}$$

Whence at length substituting these Values of the Coefficients f, b, k, l, m, n , in the assumed Equation $x = fz + bz^2 + kz^3 + lz^4 + mz^5 + nz^6$,

&c. and the Root sought will be $x = \frac{z}{a} - \frac{bz^2}{a^3}$

$$+ \frac{2b^2 - ac}{a^5} z^3 + \frac{5abc - 5b^3 - a^2d}{a^7} z^4 +$$

$$\frac{14b^4 + 6a^2bd - 21ab^2c + 3a^2c^2 - a^3e}{a^9} z^5, \&c.$$

Note, If there are any Terms wanting in the proposed Equation, it is plain that they will

likewise be wanting in the Root. For Example: If z be $= ax + cx^3 + ex^5$, &c. then will

$$x = \frac{z}{a} - \frac{ac}{a^3} z^3 + \frac{3a^2c^2 - a^3e}{a^9} z^5, \text{ \&c.}$$

In like manner, if z be $= ax + bx^3 + cx^5 + dx^7 + ex^9$, &c. then on the contrary will x be $=$

$$\frac{z}{a} - \frac{b}{a^4} z^3 + \frac{3bb - ac}{a^7} z^5 + \frac{8abc - aad - 12b^3}{a^{10}} z^7 + \frac{55b^4 - 55abbc + 10aabd + 5aacc - a^3e}{a^{13}} z^9,$$

&c.

SCHOLIUM.

3. **T**HESSE two Expressions for the Root x being thus found, will now serve as Canons for finding the Root of a proposed infinite Equation by Substitution.

For Example: If you would extract the Root of this Equation $z = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$, &c. Then substituting in the first Expression 1 for a , $-\frac{1}{2}$ for b , $\frac{1}{3}$ for c , $-\frac{1}{4}$ for d , and $\frac{1}{5}$ for e ; there will come out $x = z + \frac{1}{2}zz + \frac{1}{8}z^3 + \frac{1}{24}z^4$, &c. Moreover, the Root of this Equation $z = x - \frac{x^3}{3r} + \frac{x^5}{5r^4} - \frac{x^7}{7r^6} + \frac{x^9}{9r^8}$, &c. will be $x = z + \frac{z^3}{3r^2} + \frac{2z^5}{15r^4} + \frac{17z^7}{315r^6} + \frac{62z^9}{2835r^8}$, &c. by putting in the Expression 1 for a , $-\frac{1}{3r}$ for b , $\frac{1}{5r^4}$ for c , $-\frac{1}{7r^6}$ for d , $\frac{1}{9r^8}$ for e , &c.

SECT.



SECTION II.

Of finding the Fluents or flowing Quantities of given fluxionary Expressions.

DEFINITION.

THE fluent or flowing Quantity of a given fluxionary Expression, is that Quantity whereof the given fluxionary Expression is the Fluxion. As the Fluent of $\dot{x}$ is x , the Fluent of $\dot{x} + \dot{y}$ is $x + y$, the Fluent of $\dot{x}y + y\dot{x}$ is xy , the Fluent of $m\dot{x}x^{m-1}$ is x^m , the Fluent of $a\dot{x}x^{\frac{m}{n}}$ is $\frac{an}{m+n}x^{\frac{m+n}{n}}$.

COROL. I.

4. **H**ENCE if the Ordinate $PM(y)$ of a Curve (or strait Line) AM at right Angles to the Absciss $AP(x)$ drawn into $Pp(\dot{x})$ represents any given fluxionary Expression; then the Area of the Space APM will be the fluent or flowing Quantity of the given fluxionary Expression; and, *vice versa*, the Rectangle under the Ordinate PM , and $\dot{x}$ the Fluxion of the Absciss AP , will be the Fluxion of the Area or Space APM . For this Rectangle may be taken for the Trapezium $PMmp$, which

FIG. 1.

is

is the real Fluxion of that Area: because their Difference is only the small Triangle Mmn , infinitely less than $PMmp$. And so it may be rejected. (*Axiom I. Part I.*)

COROL. II.

HENCE likewise it appears that the inverse Method of Fluxions is a kind of a general Way of summing up of Series.

SCHOLIUM.

6. **H**ERE we may observe, that a Fluent can have but one Fluxion; but on the contrary, a Fluxion may have an infinite Number of Fluents. For Example: The Fluent ax will have but only this Fluxion $a\dot{x}$. And if $b, c, d, f, g, \&c.$ be constant Quantities; then will $ax \pm b, ax \pm c, ax \pm d, ax \pm f, ax \pm g$: or $ax \pm$, an infinite Variety of other constant Quantities, be each the Fluent of $a\dot{x}$. Whence accurately speaking, the Fluent or flowing Quantity of $a\dot{x}$ is not ax , but $ax \pm p$. p being any given Quantity, which may be equal to any other Expression whatsoever, consisting of constant Quantities. The same may be understood of the Fluents of other fluxionary Expressions.

As it is easy to raise a given Quantity to any given Power; but on the contrary, any Root thereof cannot be had in finite Terms; so likewise in the Business of Fluxions, it is easy to find the Fluxion of any variable Quantity, or variable and constant Quantities any how compounded together. But on the contrary, the Fluent of any given Fluxion cannot be had in finite

finite Terms. For as in Algebra we have recourse to Approximations in the Extraction of Surd Roots, where they cannot be exactly express'd; so in the Inverse Method of Fluxions we make use of infinite Series, where Fluents cannot be had exactly.

P R O B. I.

7. **T**O find the Fluent of a given fluxionary Expression.

Case I. 1. When fluxionary Expressions consist of no Powers of flowing or variable Quantities, but Products of flowing Quantities multiply'd by Fluxions, and also the Fluxion of every flowing Quantity that is in the Expression, as $\dot{y}x + \dot{x}y$, or $xyz + zx\dot{y} + zy\dot{x}$. The Fluents are had by this Rule, which is the Reverse of the direct Operation, viz.

Instead of each Fluxion substitute its respective variable Quantity; and adding all the Terms together, divide that Sum by the Number of Terms.

So the Fluent of $\dot{y}x + \dot{x}y$ will be xy , and of $xyz + zx\dot{y} + zy\dot{x}$ will be xyz .

2. When simple fluxionary Expressions, involving some Power of the variable Quantity, occur, viz. multiplied into some standing Quantity, as $2x\dot{x}$, or $3xx\dot{x}$, or $mx^{m-1}\dot{x}$, or $\frac{n}{m}x^{\frac{n-m}{m}}\dot{x}$, or $ax^{\frac{m}{n}}\dot{x}$, which is the most general Expression of all of this Nature: The Fluent will be had by the Reverse of the direct Operation. For as in the direct Operation, any of the foregoing Expressions is found by lessening the Index of the Power of the variable Quantity by 1, putting in the fluxionary

nary Letter $\dot{x}$, and multiplying the whole by the Index of the Power of the variable Quantity; so we come back again to the Fluent by adding 1 to the Index of the Power of the variable Quantity, striking out the fluxionary Letter $\dot{x}$, and dividing all by the Exponent thus increas'd by 1: therefore in all such Cases this is the Rule.

Strike out the fluxionary Letter, add Unity to the Exponent of the variable Quantity in the Expression, and divide it by that Exponent thus increas'd by Unity.

Hence the Fluent of $2x\dot{x}$, or $2x^1\dot{x}$ is x^2 . For striking out $\dot{x}$, and adding 1 to 1 (the Exponent of x in the given Expression) there will be had $2x^2$; which divided by 2 (= the Exponent increased by 1) the Quotient x^2 will be the Fluent of $2x\dot{x}$.

In like manner the Fluent of $3x^2\dot{x}$ will be x^3 ; of $mx^{m-1}\dot{x}$ will be x^m ; of $\frac{n}{m}x^{\frac{n-m}{m}}\dot{x}$ will be $x^{\frac{n}{m}}$, of $-nx^{-n-1}\dot{x}$ will be x^{-n} or $\frac{1}{x^n}$; of $ax^{\frac{m}{n}}\dot{x}$ will be $\frac{an}{m+n}x^{\frac{m+n}{n}}$. For in this latter Case by adding 1 to $\frac{m}{n}$ the Index of the Power of the variable Quantity, and striking out the fluxionary Letter $\dot{x}$, we have $ax^{\frac{m+n}{n}}$; which divided by $\frac{m+n}{n}$ (the new Index) and the Quotient is $\frac{a}{m+n}x^{\frac{m+n}{n}}$, viz. the Fluent of $ax^{\frac{m}{n}}\dot{x}$.

Other-

Otherwise:

Because this latter Fluxion and its Fluent are the most general of any of those of the above-named Condition: They may serve as a Canon for finding of Fluents of such simple fluxionary Expressions as abovesaid, by bringing them under the same Form, and afterwards substituting. For Example: To find the Fluent of $x^2 \dot{x}$. Now this brought to the same Form with the Fluxion $a x^{\frac{m}{n}} \dot{x}$ will be $1 x^{\frac{2}{1}} \dot{x}$; so that $a = 1$, $n = 1$, and $m = 2$. Whence putting 1 for a , and 1 for n , and 2 for m in the Fluent $\frac{a n}{m+n} x^{\frac{m+n}{n}}$, and then we shall have $\frac{1}{3} x^3$ for the Fluent of $1 x^2 \dot{x}$.

In like manner the Fluent of $4\sqrt{x} \dot{x} (= 4x^{\frac{1}{2}} \dot{x})$ will be $\frac{8}{3} x^{\frac{3}{2}} (= \frac{8}{3} \sqrt{x^3})$.

The Fluent of $\sqrt[3]{x^5} \dot{x} (= x^{\frac{5}{3}} \dot{x})$ will be $\frac{3}{8} x^{\frac{8}{3}} = \frac{3}{8} \sqrt[3]{x^8}$.

The Fluent of $\frac{1}{x^3} \dot{x} (= 1 x^{-3} \dot{x})$ will be $(\frac{-1}{-3} x^{-\frac{3}{2}} =) -x^{-\frac{1}{2}}$. For here $a = 1$, $n = 3$, and $m = -2$.

The Fluent of $\frac{1}{\sqrt{x^3}} \dot{x} (= x^{-\frac{3}{2}} \dot{x})$ will be $(\frac{-2}{-1} x^{-\frac{1}{2}} =) \frac{2}{\sqrt{x}}$.

Lastly, The Fluent of $\frac{1}{x} \dot{x} (= 1 x^{-1} \dot{x})$ will

U

will be $\frac{1}{2}x^{\frac{1}{2}} = \frac{1}{2}x^0 = \frac{1}{2} \times 1 = \frac{1}{2} = \text{Infinity}$. For here $a = 1$, $m = -1$, and $n = 1$.

Case II. If a fluxionary Expression consists of any Number of simple Terms, such as those in *N. 2. Case I.* the Fluent thereof will likewise consist of the Fluents of the several Terms of the Expression connected together with the Signs $+$ and $-$. For Example:

The Fluent of $x^2\dot{x} + x^{\frac{3}{2}}\dot{x}$ or $x^2 + x^{\frac{3}{2}} \times \dot{x}$, will be $\frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}}$. For (by *N. 2. Case I.*) the Fluent of $x^2\dot{x}$ will be $\frac{1}{3}x^3$, and that of $x^{\frac{3}{2}}\dot{x}$ will be $\frac{2}{5}x^{\frac{5}{2}}$. And so the Sum of these Fluents will be the Fluent of the Sum of those Fluxions.

So likewise the Fluent of $x^2\dot{x} - x^{\frac{3}{2}}\dot{x}$ will be $\frac{1}{3}x^3 - \frac{2}{5}x^{\frac{5}{2}}$.

And that of $3x\dot{x} - 2x^2\dot{x} + x^3\dot{x} - 5x^4\dot{x}$ will be $\frac{3}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 - x^5$.

Moreover, that of $x^{-2}\dot{x} + x^{-\frac{3}{2}}\dot{x}$, or that of $x^{-2}\dot{x} - x^{-\frac{3}{2}}\dot{x}$, will be $-x^{-1} - 2x^{-\frac{1}{2}}$, or $-x^{-1} + 2x^{-\frac{1}{2}}$; and by changing the Signs we shall have the affirmative Values, viz. $x^{-1} + 2x^{-\frac{1}{2}}$, or $x^{-1} - 2x^{-\frac{1}{2}}$.

That of $x^2\dot{x} + x^{-2}\dot{x}$, will be $\frac{1}{3}x^3 - x^{-1}$.

Case III. When the Term or Terms of a Fluxion are more compounded than any of those foregoing, they must first be reduc'd to simple Terms like some of those of the foregoing *Cases*, by throwing the Expression, or some

some Part of it, into an infinite Series, according to the Rules of the first Section: then the Fluent of the Expression, thus brought into an infinite Series of simple Terms, may be had by the Rules before laid down. For Example: Let $\frac{b}{a+x} \dot{x}$ be a given Fluxion.

Now this thrown into an infinite Series (by *Prob. 1. Sect. 1.*) will be $\frac{b}{a} \dot{x} - \frac{bx}{a^2} \dot{x} + \frac{bx^2}{a^3} \dot{x} - \frac{bx^3}{a^4} \dot{x}$, &c. the Fluent of which by *Case 2.*

will be $\frac{b}{a} x - \frac{b}{2a^2} x^2 + \frac{b}{3a^3} x^3 - \frac{bx^4}{4a^4}$, &c.

In like manner the Fluent of $\frac{1}{1+xx} \dot{x} = * * \text{ Art. 1.}$
 $1 \dot{x} - x^2 \dot{x} + x^4 \dot{x} - x^6 \dot{x}$, &c. will be $* x - \frac{1}{3} x^3 * \text{ Art. 7.}$
 $+\frac{1}{5} x^5 - \frac{1}{7} x^7$, &c.

The Fluent of $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x} \dot{x} = * 2x^{\frac{1}{2}} \dot{x} - 2x \dot{x} * \text{ Art. 1.}$
 $+ 7x^{\frac{3}{2}} \dot{x} - 13x^2 \dot{x} + 34x^{\frac{5}{2}} \dot{x}$, &c. will be $* \frac{4}{3} x^{\frac{3}{2}} * \text{ Art. 7.}$
 $- x^2 + \frac{1}{5} x^{\frac{5}{2}} - \frac{1}{7} x^{\frac{7}{2}}$, &c.

Also the Fluent of $\sqrt{aa+xx} \times \dot{x} = * a \dot{x} + * \text{ Art. 2.}$
 $\frac{x^2}{2a} \dot{x} - \frac{x^4}{8a^3} \dot{x} + \frac{x^6}{16a^5} \dot{x} - \frac{5x^8}{128a^7}$, &c. will be
 $* ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$, &c. $* \text{ Art. 7.}$

The Fluent of $\frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}}$ will be
 $x + \frac{1}{2}b \left. \begin{matrix} x^3 + \frac{3}{4}b^2 \\ + \frac{1}{2}ab \\ - \frac{1}{4}a^2 \end{matrix} \right\} x^5, \text{ \&c.}$

By the same Rules the Fluent of $\overline{dx^m \times e + f x^n \times x}$ may be found: Where d, e, f are any given Quantities, and m, n, p the Indices of the Powers of the Quantities to whom they are affixed.

For make $\frac{m+1}{n} = r, p + r = s, \frac{d}{nf}$

$\times \overline{e + f x^{p+1}} = Q$, and $rn - n = t$: then will

the Fluent be $Q \times \frac{x^t}{s} - \frac{r-1}{s-1} \times \frac{eA}{fx^n} + \frac{r-2}{s-2} \times \frac{eB}{fx^n}$

$- \frac{r-3}{s-3} \times \frac{eC}{fx^n} + \frac{r-4}{s-4} \times \frac{eD}{fx^n}, \&c.$ The Let-

ters $A, B, C, D, \&c.$ expressing the nearest preceding Terms, viz. A the Term $\frac{x^t}{s}, B$ the Term

$- \frac{r-1}{s-1} \times \frac{eA}{fx^n}, \&c.$ This Series when r is a

Fraction or negative Number will not terminate, that is, the Fluent will consist of an infinite Series of Terms. But when r is an affirmative whole Number, the Fluent will consist of a finite Number of Terms, viz. so many as there are Units in r .

Otherwise.

This last Fluxion and its Fluent will serve as a Canon for finding the Fluents of fluxionary Expressions any how compounded, not exceeding Binomials, by bringing them to the same Form with the fluxionary Expression, and afterwards by Substitution, as you may see in the following Examples.

EXAMPLE I.

To find the Fluent of $\sqrt{ax} \times x$. This brought to the same Form with the fluxionary Expression

pression above, and then $1x^0 \times 0 + ax^1 \dot{x} = dx^m$
 $\times e + fx^n \dot{x}$. Whence $d=1, m=0, e=0, f=a,$
 $n=1, p=\frac{1}{2}, Q = \frac{1}{a} \times ax^{\frac{3}{2}}, t=0, r=1,$
 $s=1\frac{1}{2}$. And so by substituting these Va-
 lues for their Equals in the general Fluent a-
 bove, and we shall have $\frac{1}{a} \times ax^{\frac{3}{2}} \times \frac{1}{1\frac{1}{2}} = \frac{2}{3}x\sqrt{ax}$
 the Fluent sought. And generally the Fluent
 of $cx^n \dot{x}$ will be $\frac{c}{n+1} x^{n+1}$.

EXAMPLE II.

$\frac{a^4x}{c^4-2ccxx+x^4} \dot{x}$ brought to the Form of the
 general Binomial fluxionary Expression will be
 $\frac{a^4x \times cc - xx}{c^4-2ccxx+x^4}$, or $\frac{a^4x^{-3}x - 1 + ccx^{-2}}{c^4-2ccxx+x^4} =$
 $\frac{dx^m \times e + fx^n}{c^4-2ccxx+x^4}$. In the first Case, $d=a^4, m=1,$
 $e=cc, f=-1, n=2, p=-2$. Whence
 $r=1, s=-1, Q = -\frac{a^4}{2} \times cc - xx^{-1}$, that is,
 $-\frac{a^4}{2cc-2xx}, t=0$. And the Fluent will be
 $Q \times -\frac{x^0}{1}$, that is, $= -\frac{a^4}{2cc-2xx}$. In the se-
 cond Case, d is $=a^4, m=-3, e=-1, f=cc,$
 $n=-2, p=-2, r=1, s=-1, Q = -\frac{a^4}{2cc}$
 $\times -1 + ccx^{-2}$, that is, $= -\frac{a^4xx}{2c^4-2ccxx}, t=0$.
 And the Fluent will be $Q \times -\frac{x^0}{1}$, that is,
 $\frac{a^4xx}{2c^4-2ccxx}$.

EXAM-

EXAMPLE III.

$\frac{a^5}{x^4} \sqrt{bx+xx} \dot{x}$ brought to the general Form will be $a^5 x^{-\frac{9}{2}} \times b + x^{\frac{1}{2}} \times \dot{x}$, or $a^5 x^{-4} \times \frac{1}{1+bx^{-1}} \times \dot{x}$. In the first Case d will be $=a^5$, $m=-\frac{9}{2}$, $e=b$, $f=1$, $n=p=\frac{1}{2}$. And so $r=-\frac{7}{2}$, &c. Now since r is negative, I try the other Case. Here d is $=a^5$, $m=-4$, $e=1$, $f=b$, $n=-1$, $p=\frac{1}{2}$. And so $r=3$, $s=3\frac{1}{2}$, $\mathcal{Q}=-\frac{a^5}{b} \times \frac{1}{1+bx^{-1}} \times \frac{1}{2}$ or $-\frac{a^5 x + a^5 b}{bxx} \sqrt{xx+bx}$, and $t=-2$. Whence the Fluent will be $\mathcal{Q} \times \frac{x^{-2}}{3\frac{1}{2}} - \frac{2}{2\frac{1}{2}} \times \frac{x^{-1}}{3\frac{1}{2}b} + \frac{1}{1\frac{1}{2}} \times \frac{2}{2\frac{1}{2}} \times \frac{x^0}{3\frac{1}{2}bb}$, that is, $-\frac{20bb+24bx-16xx}{105bbxx} \times \frac{a^5 x + a^5 b}{bxx} \sqrt{xx+bx}$.

EXAMPLE IV.

$\frac{bx^{\frac{1}{2}}}{\sqrt{c^3-3accx^{\frac{2}{3}}+3aacx^{\frac{4}{3}}-a^3xx}} \dot{x}$, brought to the Form as above, will be $bx^{\frac{1}{2}} \times c - ax^{\frac{2}{3}} \times \dot{x}$. And so $d=b$, $m=\frac{1}{2}$, $e=c$, $f=-a$, $n=\frac{2}{3}$, $p=-\frac{2}{3}$, $r=2$, $s=\frac{7}{2}$, $\mathcal{Q}=-\frac{2b}{2a} \times c - ax^{\frac{2}{3}}$, $t=\frac{2}{3}$. Then the Fluent will be $\mathcal{Q} \times \frac{x^{\frac{12}{3}}}{7} - \frac{5}{2} \times -\frac{5c}{7a}$, that is, $-\frac{20abx^{\frac{2}{3}}+75bc}{28aa} \times c - ax^{\frac{2}{3}}$.

SCHO-

SCHOLIUM I.

8. **G**ENERAL Forms may be found for the Fluents of Trinomial, or other more compounded fluxionary Expressions, after the same manner as the general Form above; and they used for finding the Fluents of fluxionary Expressions proposed, which are less compounded, or can be brought to the same Form with them. But the Work this way being most commonly extremely tedious, it will be best to get the Fluents by (the first way of *Case 3.* aforegoing) bringing those compound Terms to infinite Series of single Terms.

But compound Expressions must not be thrown into infinite Series before we have tried to reduce them by augmenting, lessening, multiplying, dividing, &c. the variable Quantities: For by this means they may often be brought down to such simple Forms as come under *Case 2.* aforegoing, and the Fluents of them be had in finite Terms.

Here it will be of Use to observe likewise, that if in radical fluxionary Expressions, the rational Part of the Expression, or that without the *Vinculum*, multiplied into the Fluxion of the variable Quantity, be the Fluxion of the Part under the *Vinculum*, or in some given Ratio to it; the Fluent will always be had in finite Terms, by *Case 2.* aforegoing, and by Substitution.

EXAMPLE. I.

THE Fluent of $ax \sqrt{ax - aa}$ or $ax - aa \times ax^{\frac{1}{2}}$,
 where ax is the Fluxion of $\sqrt{ax - aa}$ or
 $ax - aa^{\frac{1}{2}}$ will (by *Case 2.*) be $\frac{2}{3} ax - aa^{\frac{3}{2}} =$
 $\frac{2}{3} ax - aa \sqrt{ax - aa}.$ EXAM-

EXAMPLE II.

THE Fluent of $2x\dot{x}\sqrt{xx+aa}$ or $xx+aa$ ^{$\frac{1}{2}$}
 $\times 2x\dot{x}$, where $2x\dot{x}$ is the Fluxion of
 $xx+aa$ ^{$\frac{1}{2}$} , will be $\frac{2xx+2aa}{3\sqrt{xx+aa}}$

EXAMPLE III.

THE Fluent of $a+x^{\frac{m}{n}}x\dot{x}$ will be
 $\frac{n}{m+n}a+x^{\frac{m+n}{n}}$.

EXAMPLE IV.

THE Fluent of $x\dot{x}\sqrt{xx+aa}$. Where the
 Fluxion $x\dot{x}$ without the *Vinculum* is to
 the Fluxion of the Quantity under it, viz.
 $2x\dot{x}$ as 1 to 2, will be $\frac{1}{2}xx+\frac{1}{2}a\times\sqrt{xx+aa}$.
 For make $\sqrt{xx+aa}=z$: then $2z\dot{z}=2x\dot{x}$;
 and so $\sqrt{xx+aa}\times x\dot{x}=z^2\dot{z}$. Whence the
 Fluent (by *Case 2.*) $=\frac{1}{3}z^3=$ by Substitution
 to the aforesaid Fluent. After this manner
 might the Fluent of the second Example have
 been found by putting $xx+aa=z^2$.

EXAMPLE V.

THE Fluent of $x^m+a^n\times x^{m-1}\dot{x}$. Where
 the Fluxion $x^{m-1}\dot{x}$ without the *Vincu-*
lum is to the Fluxion of the Quantity con-
 tained under it, viz. $mx^{m-1}\dot{x}$, as 1 to m , will
 be $\frac{1}{mn+m}x^m+a^n$ ^{$n+1$} . For put $x^m+a^n=z$:

then

then will $\overline{x^m + aq}^n = z^n$, and $\overline{x^m + aq}^{n-1} = z^{n-1}$. Also $nz^{n-1} \dot{z} = nm\overline{x^m + aq}^{n-1} \dot{x} \times \overline{x^m + aq}^{n-1} = nm\overline{x^m + aq}^{n-1} \dot{x} \times z^{n-1}$. And dividing by nz^{n-1} , there comes out $\dot{z} = m\overline{x^m + aq}^{n-1} \dot{x}$, or $\overline{x^m + aq}^{n-1} \dot{x} = \frac{\dot{z}}{m}$. Whence $\overline{x^m + aq}^n \dot{x} = \frac{z^n \dot{z}}{m}$; and

$$\text{the Fluent} = \frac{1}{nm + m} z^{n+1} = \frac{1}{nm + m} \overline{x^m + aq}^{n+1}.$$

Here it may not be amiss to give the following Table of simple Fluxions, (from Sir ISAAC NEWTON's *Quadrature of Curves*) whose Fluents standing against them are expressed in finite Terms. By which Means the Fluent of a Fluxion coming under any of the Forms of the Fluxions in the said Table may be had by Inspection, or else a very easy Substitution. Here z is the variable Quantity, and d, e, f, g, h, u are invariable or given Quantities.

X

Forms

Forms of Fluxions.		Fluents.
I.	$\dot{x}x^n-1$	Flu. $= \frac{d}{n} x^n.$
II.	$\frac{\dot{x}x^{2n}-1}{e+2efx^n+f^2x^{2n}}$	Flu. $= \frac{dx^n}{ne^2+nefx^n}$, or $\frac{-d}{nef+nf^2x^n}.$
III.	1. $\dot{x}x^{2n}-1 \sqrt{e+fx^n}$	Flu. $= \frac{2d}{3nf} R^3$, R being $= \sqrt{e+fx^n}.$
	2. $\dot{x}x^{2n}-1 \sqrt{e+fx^n}$	Flu. $= \frac{-4e+6fx^n}{15nf^2} dR^3.$
	3. $\dot{x}x^{3n}-1 \sqrt{e+fx^n}$	Flu. $= \frac{16e^2-24efx^n+30f^2x^{2n}}{105nf^3} dR^3.$
	4. $\dot{x}x^{4n}-1 \sqrt{e+fx^n}$	Flu. $= \frac{-96e^3+144e^2fx^n-180ef^2x^{2n}+210f^3x^{3n}}{945nf^4} dR^3.$
IV.	1. $\frac{\dot{x}x^{2n}-1}{\sqrt{e+fx^n}}$	Flu. $= \frac{2d}{nf} R.$
	2. $\frac{\dot{x}x^{2n}-1}{\sqrt{e+fx^n}}$	Flu. $= \frac{-4e+2fx^n}{3nf^2} dR.$
	3. $\frac{\dot{x}x^{3n}-1}{\sqrt{e+fx^n}}$	Flu. $= \frac{16e^2-8efx^n+6f^2x^{2n}}{15nf^3} dR.$
	4. $\frac{\dot{x}x^{4n}-1}{\sqrt{e+fx^n}}$	Flu. $= \frac{-96e^3+48e^2fx^n-36ef^2x^{2n}+30f^3x^{3n}}{105nf^4} dR.$

S C H O L I U M II.

BEFORE I conclude this Section, I will take the Liberty of adding a Word or two concerning the excellent compendious Method of expressing the Fluents of given Fluxions by the Measures of Ratio's and Angles

gles written by the late Mr. *Cotes*, Professor of Astronomy and Experimental Philosophy in the University of *Cambridge*; and publish'd after his Death by his Successor Dr. *Smith*, under the Title of *Harmonia Mensurarum*.

Here the Labour of throwing Quantities into infinite Series, which in many Cases is very troublesome, and on account of their converging sometimes too slowly are not fit for Use, is entirely avoided, and elegant Constructions of the Fluents of Fluxions are had geometrically with the Assistance of ample Tables of Logarithms of *Briggs's Form*, for finding the Measures of Ratio's, and of large Tables of natural Sines and Tangents for finding the Measures of Angles. And from hence may be deduced wonderful neat and compendious Solutions of all difficult Problems; such as the Quadrature of Curve-lin'd Spaces, Rectification of Curves, Cubation of Solids, &c. wherein the Fluents of given Fluxions are concerned. Several Examples of which I shall give hereafter.

In the Treatise before us, you have two Series of Tables of several Forms of Fluxions at the Head of each Page, with their Fluents underneath them expressed in the Measures of Ratio's or Angles; the one Series composed by Mr. *Cotes* himself, and the other by Dr. *Smith*. In Mr. *Cotes's* Tables, which I shall confine myself to, they being sufficient for Purposes that usually occur, z is the variable Quantity, d, e, f standing Quantities, n any Index of the Power of z , $\pm$ any affirmative or negative Number; and the Quantities R, S, T , always being the three Sides of a right-angled Triangle, whose Values are set down

at the Bottoms of each Page, express the Ratio or Angle, by the Measure of which the Fluent of the given Fluxion is had: and if R be the square Root of an affirmative Quantity, they express a Ratio; always being that of $R + T$ to S . But if R be the square Root of a negative Quantity they denote an Angle, which Angle will be always that whose Tangent and Secant are to the Radius as T and S to R , where the Sign of that negative Quantity is changed into an affirmative one. The Figures in the Column of each Page of the Tables, with θ at the top, are some of the affirmative and negative Values of θ , against which the Fluents of the fluxionary Forms a top stand. As in the second Form $\frac{dz z^{\theta n + \frac{1}{2}n - 1}}{e + fz^n}$.

When θ is 2, the Fluent will be $\frac{2 dz z^{\frac{3}{2}n}}{3nf}$

$-\frac{2dez^{\frac{1}{2}n}}{nff} + \frac{2e}{nff} dR \left| \frac{R+T}{S} \right.$. When θ is

0, the Fluent will be $\frac{2}{ne} dR \left| \frac{R+T}{S} \right.$. And

when θ is -1 , the Fluent will be $\frac{-2d}{nez^{\frac{1}{2}n}} + \frac{2}{ne} dr \left| \frac{r+t}{s} \right.$. And so of others.

But these Fluents cannot be said to be entirely known till the Quantity $R \left| \frac{R+T}{S} \right.$ be found: Which is called the Measure of the Ratio of $R+T$ and S to the *Module* R , when R is affirmative, or till the Quantity $\frac{2e}{nff} dR \left| \frac{R+T}{S} \right.$, or $\frac{2}{ne} dR \left| \frac{R+T}{S} \right.$, or $\frac{2}{ne} dr \left| \frac{r+t}{s} \right.$ in

in the several Fluents be found; the first of which is the Measure of the Ratio of $R + T$ and S to the *Module* $\frac{2e}{nff}dR$; the second the Measure of the Ratio of $R + T$, and S to the *Module* $\frac{2d}{ne}R$; and the third of $R + T$ and S to the *Module* $\frac{2}{ne}dR$. Or when R is nega-

tive, the Quantity $R \left| \frac{R+T}{S} \right|$ is = Measure of an Angle, whose Radius, Tangent and Secant are the respective Values of R , T and S , to the Radius R as a *Module*. And the Way to find this Measure of a Ratio or Angle to a given *Module* I shall shew presently; but first take the following Definitions, or Descriptions of Terms, in order to clear this so far, as that a Person may in some measure understand the Use of these excellent Tables, without being at the pains of reading the Propositions in the first Part of the *Harmonia Mensurarum*: which are too generally handled to be perceived by a moderate Capacity without much Application.

DEFIN. I.

THE *Measure* of a Ratio is any Quantity proportional to that Ratio; that is, if M be the Measure of the Ratio of A to B , or of $\frac{A}{B}$, and m the Measure of the Ratio of a to b , or $\frac{a}{b}$; then will $M : \frac{A}{B} :: m : \frac{a}{b}$. Therefore equal Ratio's have the same Measure. If one Ratio be the Double of the other, the Measure

Measure of the former will be the Double of the Measure of the latter: if the former be the Triple of the latter, the Measure of the former will be the Triple of the Measure of the latter: if the Half, the Half, &c. So that if the Ratio be never so much increased or lessen'd by Composition or Resolution, the Measure thereof will be likewise increased or lessen'd proportionably.

Moreover the Measure of a Ratio of Equality is 0; and if the Measure of the Ratio of a greater Quantity to a less be supposed positive, then the Measure of the Ratio of a lesser Quantity to a greater will be negative.

DEFIN. II.

THE *Numerical Measure of a Ratio* is the Excess of the Logarithm of a Number expressing the Antecedent above the Logarithm of the Number expressing the Consequent; that is, the Logarithm of the Quotient of the Division of the Antecedent by the Consequent, is the numerical Measure of a numerical Ratio.

DEFIN. III.

THE *Trigonometrical Measure of an Angle* is the Quantity of Degrees, Minutes, Seconds, &c. contained in that Angle.

DEFIN. IV.

THE *Module of Briggs's, Vlaque's, &c. Logarithms* is 0,434294481903, &c, by which if you divide 1, the Quotient 2,302585092994, &c. will be the reciprocal
Module

Module of the said Logarithms; that is, the Quotient of the Division of any Quantity by the first *Module*, is equal to the Product of that Quantity multiplied by the reciprocal *Module*.

This Definition, or rather Account of the Quantity of the *Module* of the Logarithms, is sufficient for my Purpose. Those who are not satisfy'd with it may consult *Prop. 1.* and its *Corollaries*, and *Scholia* in Part 1. *Harmonia Mensurarum*. The same may be said likewise of the following Definition, which is a Consequence of the *Prop.* in the Notes of the ingenious Dr. *Smith* contained in p. 94. at the latter End of *Harmonia Mensurarum*.

DEFIN. V.

THE *Module* of the Trigonometrical Canon, or the Number of Degrees contained in the Arch of a Circle equal to the Radius, which is to 180 Degrees, as the Radius of a Circle to $\frac{1}{2}$ the Circumference, is $57^{\circ} 17' 44''$, or 57,2957795130; and dividing 1 by it, the reciprocal *Module* of the said Canon will be 0,0174532925.

PROP. I.

9. **T**O find the Measure of a given Ratio to a given *Module*, or to find the Quantity of the Expression $R \sqrt{\frac{R+T}{S}}$ when *R* is the square Root of an affirmative Quantity, and *R*, *T*, *S*, the three Sides of a right-angled Triangle.

Rule. As the *Module* of the Logarithms 0,434294481903, &c. is to the *Module* *R* of the

the Ratio $R + T$ to S ; so is the Logarithm of this Ratio to the Measure of it, having R as a *Module*, which will be equal to $R \left| \frac{R+T}{S} \right.$. or multiply the Product of the Logarithm of the proposed Ratio of $R + T$ to S into the Quantity R , as a *Module*, by the reciprocal *Module* 2.302585092994, &c. and this second Product will be the Measure of the Ratio of $R + T$ to S with R , being the *Module* thereof; and it is equal to the Value of the Quantity $R \left| \frac{R+T}{S} \right.$ sought.

Take this Numerical Example. Let $R=8$, $T=6$, $S=10$.

Module of the Logarithm 0,434294481903 : 8 :: Logarithm (of $\frac{14}{10}$) 0,1461280 : 2,6916777 = Measure of the Ratio of 14, and 10 to the *Module* 8, equal to $8 \left| \frac{8+6}{10} \right.$.

Or shorter thus: 1 : recip. Mod. Logarithm 2.302585092994 :: Logarithm (of $\frac{14}{10}$) 0,1461280 $\times$ 8 the given *Module* : 2,6916777 the Measure of the Ratio as before.

FIG. 2. This Problem may be solved without Computation by means of the Sector of an Hyperbola, after the following manner. Let AG be an Hyperbola, CA the Semi-transverse Diameter, and CB the Semi-conjugate, and CE an Asymptote: and draw AQ parallel to CB . Then make $R:T::AQ:AD$. And if $CA \times CB$ be $= 2R$, the Sector CAM will be $= R \left| \frac{R+T}{S} \right.$. The Triangle CAD being equal to T , when T is less than R ; and the Triangle

gle CBE equal to T , when T is greater than R .

COROL. I.

10. **H**ENCE if m be the standing *Module* of the Logarithms, and l the Logarithm of the Ratio $\frac{R+T}{S}$, then will $R \left| \frac{R+T}{S} = R \times \frac{l}{m} \right.$

COROL. II.

11. **H**ENCE $\frac{nR}{m} \left| \frac{T^{\frac{m}{n}}}{S^{\frac{m}{n}}} = R \left| \frac{T}{S} \right. \right.$ m being any whole Number, and n another. This follows from the Nature of the Logarithms.

PROB. II.

12. **T**O find the Measure of an Angle, whose Radius is as R , Tangent as T , and Secant as S , to the Quantity R as a Module: Or to find the Value of this Expression $R \left| \frac{R+T}{S} \right.$.
When R is the square Root of a negative Quantity, and so impossible; all of them being given Quantities.

Rule. First say, As the Value of R to the Value of T , or as the Value of R to the Value of S ; so is the Radius of the Tables of the Tangents and Secants to the Tabular Tangent or Secant of the Angle to be measur'd. Against which stands the Quantity of that Angle in Degrees, Minutes, &c. being the Trigonometrical

trical Measure of it. Then as the *Module* of the Trigonometrical Canon $57^{\circ}. 17'. 44''$. or 57,2957795130, is to the Trigonometrical Measure of the Angle just now found, so is the *Module* of that Angle, viz. R , to the Measure of the said Angle, having R as a *Module*; which is equal to the Quantity $R \left| \frac{R+T}{S} \right.$. Or mul-

tiple the Product of the Trigonometrical Measure of the Angle into the *Module* R by the reciprocal *Module* of the Canon 0,0174532925. And this second Product will be the Measure of the Angle to the given *Module* R .

Here follows a numerical Example. Let R be = 16, T = 12, and S = 20.

As $16 : 12 :: \text{Radius } 10000000 : 7500000 = \text{Tangent}$.

And as $16 : 20 :: \text{Radius } 10000000 : 12500000 = \text{Secant}$.

Against both which in the Tables you have $36^{\circ}. 52'. 6''$. for the Trigonometrical Measure of the Angle.

Again: *Module* Trigonometrical Can. 57. 2957795130: Trigonometrical Measure 36, 86833331 :: *Module* 16: 10.4330985 = Measure of the Angle, whose Radius is as 16, Tangent as 12, and Secant as 20, to the *Module* 16; or equal to $16 \left| \frac{16+12}{20} \right.$.

Note, By using the reciprocal *Module*, 0174532925 will be shorter than by using the direct *Module*: And the Operations will be still very much abbreviated, by using the Logarithms to find the fourth Terms of the Proportion, in the foregoing Problems as well as this.

This

This Problem may be solv'd likewise by means of the Sector of a Circle or Ellipsis, after the following manner. Let CA, CB , FIG. 3. the Semiaxes of the Quadrant of a Circle or Ellipsis AB , drawn into each other, be equal to $2R$. Draw AQ parallel to CB , and BG parallel to CA ; then make $R:T::CB:AD$, and draw the right Line $CMDG$; then the Sector CAM will be equal to $R \left| \frac{R+T}{S} \right|$, when R is greater than $T = \text{Triangle } CAD$, and the Sector $CBM = R \left| \frac{R+T}{S} \right|$. When R is less than $T = \text{Triangle } CBG$.

Note, When S happens to be the Square Root of a negative Quantity, and so impossible, you may change the Sign, and every thing will succeed right.

COROL.

13. IF a be the Angle; whose Radius, Tangent and Secant are as R, T, S , and m be the standing Module of the Canon; then $R \left| \frac{R+T}{S} \right|$ will be $= \frac{aR}{m}$.

SCHOLIUM.

14. BECAUSE R, T, S are always the three Sides of a right-angled Triangle; therefore $R \left| \frac{R+T}{S} \right|$ is $= R \left| \frac{S}{T-R} \right|$; supposing T to be the Hypothenufe. For from the Nature of a right-angled Triangle $T = \sqrt{RR+SS}$: and so $R \left| \frac{R+T}{S} \right|$ is $= R \left| \frac{R+\sqrt{RR+SS}}{S} \right|$. And

$R \left| \frac{S}{T-R} \right|$ is $= R \left| \frac{S}{\sqrt{RR+SS}-R} \right|$. And multiplying crosswise, you will find that $\sqrt{RR+SS} + R \times \sqrt{RR+SS} - R$ is $= SS$. The Sum of two Numbers multiplied by their Difference being equal to the Difference of their Squares; therefore $\frac{R + \sqrt{RR+SS}}{S}$ is $= \frac{S}{-R + \sqrt{RR+SS}}$; that is, $\frac{R+T}{S}$ is $= \frac{S}{T-R}$. Consequently $R \left| \frac{R+T}{S} \right| = R \left| \frac{S}{T-R} \right|$.

This evidently appears also in *Fig. 83*. For let the Hypothenuſe $AB = T$, the Perpendicular $CB = R$, and the Baſe $AC = S$. Continue out AB till $BD = BC = R$, and make $BE = BC$; then $AD = R + T$, and $AE = T - R$. If a Circle be deſcribed about B with the Radius BC , it will paſs through E and D , and AC will touch it in C ; therefore (by *Prop. 36. lib. 3. Eucl.*) $\overline{AC}^2 (SS)$ is $= AD \times AE = \overline{R+T} \times \overline{T-R}$; and ſo $\frac{R+T}{S}$ is $= \frac{S}{T-R}$.

SECT.



SECTION III.

*Use of the inverse Method of Fluxions
in the Quadrature of Curve-lined
Spaces.*

P R O B.

15. *TO square a given Curve-lined Space.*

Having gotten the Equation expressing the FIG. 1.
Relation of an Absciss $AP(x)$ to its correspond-
ent Ordinate $PM(y)$ at right Angles to each
other, find the Value of y , which multiply by
 x , and the Fluent of the Expression thus aris-
ing will express the Quadrature of the inde-
terminate mixed-line Space contained under
the Absciss AP , Ordinate PM , and Curve
 AM . And if the Absciss AP be determinate,
viz. = to a given Quantity a ; and according-
ly the Curve itself so likewise: Then by sub-
stituting a for x in the Fluent aforesaid, there
will come out an Expression for the Quadra-
ture of the determinate mixed-lined Space.
This will be plainer by the following Exam-
ples.

But if the Area $CDEF$ contained under FIG. 2.
two Curves or right Lines DE, CF , the right
Line CD , and the Part EF of any right Line
 AE drawn from a given Point A in the right
Line DC be sought, draw Afe infinitely near
 AFE , and from the Centre A describe the
small

small Arches Fp , Eq : then from the Nature of the Curves find the Area of the Quadrilateral Figure $FEqp$; which is equal to the Difference of the little Sectors AFp , AEq , or equal to $\frac{1}{2} AE \times Eq - \frac{1}{2} AF \times Fp$: and this will be = Space $FEef$, being the Fluxion of the Area $CDEF$; the Fluent of which will be = said Area. Examples of this will be shewn hereafter.

EXAMPLE I.

16. **T**O find the Area of a Triangle ABC .

FIG. 5.

Draw AD perpendicular to one of the Sides, as BC ; in which assume any where between A and D the Point P . Thro' which draw the Line MN perpendicular to AD ; and let mn be infinitely near and parallel to MN ; and draw Mp , Nq perpendicular to MN . Now the Rectangle $MNqp$, viz. under Mq , or Nq , or Pp , and the Ordinate MN will be the Fluxion of the indeterminate Area AMN . Which must first be found thus:

Call the variable Quantities AP, x ; PM, y ; and the given Quantities AD, a ; CB, b . Then because MN is parallel to AB . $AD(a) : CB(b) :: AP(x) : MN(y)$. Whence $y = \frac{bx}{a}$. But

$Pp (=Mp = Nq) = \dot{x}$. Therefore the Fluxion of the indefinite Area AMN will be $\frac{bx}{a} \dot{x}$. The Fluent of which (by Case 2.) will

be $\frac{bx^2}{2a} = \text{Area } AMN$. Now if for $AP(x)$

you substitute $AD(a)$, there will arise $\frac{ba^2}{2a}$ ($= \frac{1}{2} ab$) $= \frac{1}{2} AD \times CB$. And this will be the Area

Area of the whole Triangle ACB ; which we know likewise to be true from the Elements of Geometry.

Here it may not be amiss likewise to shew FIG. 6. the way of finding the Area of a Trapezium, $GPCB$, having two Sides PC , GB parallel, and the Angles B , C right ones, by the Inverse Method of Fluxions; tho' the thing is much shorter found out by the Elements of common Geometry. But it is delightful to perceive the same Truth arise from such very different Principles.

In order to this, continue out CB and PG to meet one another in A . From A draw Amp infinitely near AGP , and with the Distances Am , Ap describe from A the small Arches mr , pn . This being done, let $AB = a$, $BC = b$, $BG = x$, $AG = y$. Now the Triangles ACP , pPn are similar; because the Angles at n and C are right ones, and the Angle P is common to both. Whence $AG (y) : AB (a)$
 $:: Gm (\dot{x}) : mr = \frac{a \dot{x}}{y}$. This multiplied by $\frac{1}{2} Ar$

$(= \frac{1}{2} AG = \frac{y}{2})$, and the Product $\frac{a \dot{x}}{2}$ is equal to the Area of the little Triangle Arm or AGm : these differing only by the Triangle Gmr being infinitely less than either of them.

Again; Because the Triangles ABG , ACP are similar; therefore $AB (a) : AG (y) :: BC (b) : GP = \frac{by}{a}$. Consequently $AP = \frac{by}{a} + y$.

And since Am , AG , and AP , Ap differ from one another by infinitely small Quantities only: therefore Am may be taken for AG and Ap for AP . This being granted, and the Triangles Amr , Apn , taken as similar; we shall

shall have $AG(y):rm\left(\frac{a\dot{x}}{y}\right):AP\left(\frac{by}{a}+y\right):$
 $pn=\frac{b\dot{x}}{y}+\frac{a\dot{x}}{y}$; which multiplied by $\frac{1}{2}AP$
 $\left(\frac{by}{a}+\frac{y}{2}\right)$; and the Product $\frac{bb\dot{x}}{2a}+b\dot{x}+\frac{1}{2}a\dot{x}$
 is equal to the Area of the Triangle Apn or
 APp . From which if you substract the Area
 of the Triangle $AmG\left(\frac{a\dot{x}}{2}\right)$ before found, the
 Remainder $b\dot{x}+\frac{bb\dot{x}}{2a}$ will be the Area of the
 Trapezium $mpnr$, which may be taken for the
 Area of the Trapezium $mpPG$, being the Fluxion
 of the Trapezium $BCPG$. But the Fluent of this
 Fluxion is $b x + \frac{bbx}{2a} = \frac{2abx + bbx}{2a} = \frac{2ax + xb}{a}$
 $\times \frac{b}{2}$. But $\frac{ax + xb}{a}$ is $= PC$. Since from the
 Similarity of the Triangles ABG, ACP, AB
 $(a):BG(x)::AC(a+b):CP=\frac{ax+xb}{a}$.
 Therefore $\overline{GB} + PC \times \frac{1}{2}BC =$ Area of the
 Trapezium $GPCB$. Which is a known Truth
 from the Elements of common Geometry.

EXAMPLE II.

17. **T**O find the Area of (or to square) the com-
 mon parabolical Space ABD .

FIG. 1. Call the given Quantities AD, a ; BD, b :
 and the invariable Absciss AP, x ; and the cor-
 respondent Ordinate PM, y : and let Pp be $=$
 $\dot{x}$. Now the Equation expressing the Relati-
 on between $AP(x)$ and $PM(y)$ is $px=yy$.
 Whence $y=\sqrt{px}=p^{\frac{1}{2}}x^{\frac{1}{2}}$. And so the small
 Rectangle $PMnp$, equal to the Fluxion of the
 indefinite Space AMP will be $\sqrt{px}\times\dot{x}$; that
 is

is, $y\dot{x} = p^{\frac{1}{2}} \times x^{\frac{1}{2}} \dot{x}$. The Fluent of which (by *Case 2. Sect. 2.*) will be $= (\frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}} = \frac{2}{3} \sqrt{px^3} = \frac{2}{3} \sqrt{x^2 y^2}) \frac{2}{3} xy$, equal to the indeterminate Space *AMP*; and by substituting *a* for *x*, and *b* for *y* in this Fluent; and then we shall have $\frac{2}{3} ab = \frac{2}{3} AP \times PM$; that is, the parabolick Space is to the Rectangle under the Semi-ordinate and Absciss, as $\frac{2}{3} xy$ to xy , or as 2 to 3.

E X A M P L E III.

18. **T**O square Parabola's of all Kinds.

If *AP* (*x*) be the Absciss, and *PM* (*y*) the FIG. 1. correspondent Ordinate, then the Relation between the Ordinate and Absciss of a Parabola of any kind will be expressed by this general Equation $p^m x^n = y^q$. Whence $p^{\frac{m}{q}} x^{\frac{n}{q}} = y$; and so the Fluxion of the Area will be $y\dot{x} = p^{\frac{m}{q}} x^{\frac{n}{q}} \dot{x}$. The Fluent of which (by *Case 2. Sect. 2.*) will be $\frac{q}{n+q} p^{\frac{m}{q}} \frac{n+q}{xq} = \frac{q}{n+q} xy$, because $p^{\frac{m}{q}} x^{\frac{n}{q}} = y$. And so any Paraboloid is to the Rectangle under the Ordinate and Absciss, as $\frac{qxy}{n+q}$ to xy , or *q* to *n+q*.

$$x^{\frac{n+q}{q}}$$

E X A M P L E IV.

19. **T**O square the Segment of the Parabolick FIG. 7. Space *PMNQ* contained under the Ordinates *PM*, *NQ*, the Part *PQ* of the Absciss, and the Part *MN* of the Curve of the Parabola.

Z

Here

Here let $AP = a$ be invariable, and let P be the Beginning of the variable Quantity $PQ = x$. Also let QN be $= y$, and the Parameter $= p$; then AQ will be $= a + x$, draw nq parallel and infinitely near to NQ .

Now the Nature of the Curve is $AP + PQ \times p = \overline{NQ}^2$; that is, $pa + px = yy$, and $\sqrt{pa + px} = y$. Therefore $QN \times Qq = y \dot{x} = \dot{x} \sqrt{pa + px} = 1 x^0 \times pa + px^{\frac{1}{2}} \dot{x}$ is the Fluxion of the Area, the Fluent of which may be found (by the latter way of *Case 2. Sect. 2.*) or more easily thus: Let $\sqrt{pa + px} = z$; then will $pa + px = zz$, and $p \dot{x} = 2z \dot{z}$, and $\dot{x} = \frac{2z \dot{z}}{p}$: therefore $y \dot{x}$ is $= \frac{2z^2 \dot{z}}{p}$: the Fluent of

$$\text{which is } \frac{2}{3} \frac{z^3}{p} = \frac{2}{3} \frac{pa + px \times \sqrt{pa + px}}{p} = \frac{2}{3} a + x \times \sqrt{pa + px} = \frac{2}{3} A Q \times N Q.$$

But since in the Point P x is $= 0$, the Space $PMNQ$ will vanish; and so making $x = 0$ in the Fluent just now found, and the Terms x and px will vanish; so that the Fluent will be now $\frac{2}{3} a \sqrt{pa}$. Which shews what is to be added to the Fluent, that so the Space $MPNQ$ may become 0 in P ; and consequently the Quadrature of the same had. In this Case $\frac{2}{3} a \sqrt{pa}$ is to be taken from it: therefore the

$$\text{Area of } MPNQ \text{ will be } = \frac{2}{3} a + x \sqrt{pa + px} - \frac{2}{3} a \sqrt{pa} = \frac{2}{3} A Q \times N Q - \frac{2}{3} A P \times P M.$$

Otherwise.

Draw mp infinitely near to MP , and let $AQ = a$ be invariable. Let the Beginning of x be in Q , and let QP be $= x$, $PM = y$; then will AP be $= a - x$.

Now

APPENDIX.

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Now from the Nature of the Curve

$AQ - QP \times p = \overline{PM}^2$; that is, $pa - px = yy$. Whence $\sqrt{pa - px} = y$: and so $MP \times Pp$ the Fluxion of the Area will be $y \dot{x} = \dot{x} \sqrt{pa - px}$. The Fluent of which may be found as before thus: Make $pa - px = zz$. Then will $-p \dot{x}$ be $= 2z \dot{z}$, and consequently $\dot{x} = -\frac{2z^2 \dot{z}}{a}$.

Therefore $y \dot{x} = -\frac{2z^2 \dot{z}}{a}$. The Fluent of which

will be $-\frac{2z^3}{3a} = -\frac{\frac{2}{3}a - x \times \sqrt{pa - px}}{a} = \frac{\frac{2}{3}x - a}{\sqrt{pa - px}}$.

Now to find what is to be added to the Fluent to give us the Quadrature of the Space $PMNQ$, making as before $x = 0$ in the Fluent, and we have $-\frac{2}{3}a\sqrt{pa}$. Whence it is manifest, that if $+\frac{2}{3}p\sqrt{pa}$ be added to the Fluent, the Space $PMNQ$ will be $= \frac{2}{3}a\sqrt{pa} + \frac{2}{3}x - a \times \sqrt{pa - px}$.

COROL. I.

20. **T**HE Space $PMNQ = ANQ - AMP$. But in the first way $ANQ = \frac{2}{3}AQ \times QN = \frac{2}{3}a + x \times pa + px$; and $AMP = \frac{2}{3}AP \times PM = \frac{2}{3}a\sqrt{pa}$: therefore $PMNQ = \frac{2}{3}AQ \times QN - \frac{2}{3}AP \times PM$. Moreover in the latter way: $ANQ = \frac{2}{3}AQ \times QN = \frac{2}{3}a\sqrt{pa}$, and $AMP = \frac{2}{3}AP \times PM = \frac{2}{3}a - x \times \sqrt{pa - px}$: therefore $QNMP$ is $= \frac{2}{3}AQ \times QN - \frac{2}{3}AP \times PM$ here also.

Z 2

COROL.

COROL. II.

21. **I**F the Curve be not described, and the Equation expressing the Nature of it be only given, and so it is uncertain where x is to begin; it is evident from the Solution above, that 0 must be substituted for x in the Fluent; and striking out all the Terms affected with x , what is left must be added to the Fluent with the Sign changed, and the whole will be the Quadrature sought.

EXAMPLE V.

22. **T**O square a Curve expressed by this Equation $x^5 + ax^4 + a^2x^3 + a^3x^2 + a^5 = a^4y$.

Since $y = \frac{x^5}{a^4} + \frac{x^4}{a^3} + \frac{x^3}{a^2} + \frac{x^2}{a} + a$; the Fluxion of the Area will be $y \dot{x} = \frac{x^5}{a^4} + \frac{x^4}{a^3} + \frac{x^3}{a^2} + \frac{x^2}{a} + a \times \dot{x}$, and the Fluent will be $\frac{x^6}{6a^4} + \frac{x^5}{5a^3} + \frac{x^4}{4a^2} + \frac{x^3}{3a} + ax$.

EXAMPLE VI.

23. **T**O square any Curve, whose Nature is expressed by this general Equation $y = x^m \sqrt{x+a}$.

Because y is $= x \times x + a^{\frac{1}{m}}$, therefore the Fluxion of the Space sought will be $y \dot{x} = x \dot{x} \times x + a^{\frac{1}{m}}$; the Fluent of which will be easily found by making $x + a^{\frac{1}{m}} = z$. For then $x + a$

$x + a = z^m$; and both Sides of this last Equation thrown into Fluxions will be $\dot{x} = m z^{m-1} \dot{z}$.

Whence $y \dot{x} = m z^m \dot{z}$, the Fluent of which will be $= \frac{m z^m + 1}{m + 1} = \frac{m}{m + 1} \overline{x + a} \times \sqrt[m]{x + a}$.

Now in order to know whether this be the true Fluent, suppose $x = 0$; then this last Expression will become $\frac{m}{m + 1} a^m \sqrt[m]{a}$. Which must

therefore be taken from the Fluent, (by *Cor. 2. Example 3.*) so that the true Fluent or Quadrature of the Curve will be $\frac{m}{m + 1} \overline{x + a}$

$$\times \sqrt[m]{x + a} - \frac{m}{m + 1} a^m \sqrt[m]{a}.$$

EXAMPLE VII.

23. **T**O square Hyperbola's between the Asymptotes of all Kinds; or, which is the same, to find the Area of the interminate Space HMPAS, or hMPs. The former contained under the Absciss AP, Ordinate PM, Asymptote AS, and the Part MH of the Curve of the Hyperbola; and the latter under the Ordinate PM, the remaining Part Ps of the Asymptote, and the remaining Part Mh of the Curve of the Hyperbola. FIG. 8;

In these Curves the Relation between $AP(x)$ and $PM(y)$ is expressed by the following Equation $a^m + x^n = y^m x^n$.

Whence $a^m + x^n = y^m x^n$, and $a^{\frac{m+n}{n}} x^{-\frac{n}{m}} = y$.

Consequently $PM \times Pp = y \dot{x}$ is $= a^{\frac{m+n}{n}} x^{-\frac{n}{m}} \dot{x}$.

The Fluent of which will be $\left(\frac{m}{m-n} a^{\frac{m+n}{n}} x^{\frac{m-n}{n}} \right)$

$$x x^{-\frac{n+m}{m}} = \frac{m}{m-n} \sqrt[m]{a^{m+1} x^{m+n}} = \frac{m}{m-n} \sqrt[m]{y^m x^n}$$

$$= \frac{m}{m-n} x y. \text{ If } m \text{ be greater than } n, \text{ then we}$$

have always the Quadrature of the Interminate Space *HMPAS*: the same being =

$$\frac{m}{m-n} AP \times PM: \text{ If } m \text{ be less than } n, \text{ then}$$

$$\text{will } \frac{m}{m-n} x y \text{ be negative; and so we have the}$$

Quadrature of the Interminate Space *bMPs* lying on the other Side of the Ordinate *PM*.

But when $m=n$, neither of these Spaces can be squared, they being in this Case both infinite.

For if $xy^2=a^3$, then $m=2$, $n=1$; and so $HMPAS = \sqrt{a^3 x} = (\sqrt{x^2 y^2}) = xy$. If

$xy^4=a^5$, then $m=4$, $n=1$; and so $HMPAS = \frac{4}{3}xy$. If $x^2y=a^5$, then $m=1$, $n=2$; and

so $-xy$ will be the Quadrature of the Space *bMPs*. If $x^4y=a^5$, then will $m=1$, $n=4$, and so $-\frac{1}{3}xy$, that is, $\frac{1}{3}xy = bMSs$. But

when $m=n$, then $\frac{m}{m-n}$ is $=\frac{1}{0}$: therefore the

Numerator is infinite in respect of the Denominator.

EXAMPLE VIII.

25. **T**O square the common Hyperbola between its Asymptotes; or, which is the same thing, to find the Area of the Space *CcMP* contained under the Ordinates *Cc*, *PM*, the Part *CP* of the Asymptote, and the Part *cM* of the Curve of the Hyperbola.

FIG. 8.

Let the given Quantity *AC* be $=b$, and let the Beginning of x be at *C*.

Then

Then the Equation expressing the Relation of AP ($b+x$) to PM (y) will be $a^2 = by + xy$.

And so $\frac{a^2}{b+x} = y$; and $\dot{x}y$ the Fluxion of the

Area will be $\frac{a^2}{b+x} \dot{x}$.

Now the Fluent of this (by *Case 3. Sect. 1.*) will be $\frac{a^2}{b} x - \frac{a^2}{2b^2} x^2 + \frac{a^2 x^3}{3b^3} - \frac{a^2 x^4}{4b^4}$, &c. = to the Area $CcMP$. And if you suppose $a = b = 1$, then will $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. be = Area aforesaid.

Otherwise by the Measure of a Ratio or Angle.

The Fluxion $\dot{x} \frac{a^2}{b+x}$ may be referred to the 1st Form in Mr. *Cotes's* Tables. For making $z = x$, $n = 1$, $\theta = 1$, $d = a^2$, $e = b$, $f = 1$; then will

$$\frac{d \dot{z} z^{\theta n - 1}}{e + f z^n} = \frac{a^2 \dot{x}}{b + x}. \text{ And the Fluent standing}$$

$$\text{against } \theta = 1, \text{ is } \frac{d}{nf} \left| \frac{e + f z^n}{e} = a^2 \left| \frac{b + x}{b} = \overline{AC}^2 \right.$$

$$\left| \frac{AP}{AC} = \text{to the Measure of the Ratio of } AP \right.$$

and AC to $\overline{AC}^2$ for a Module, which you may find by *Art. 9.* = Area of the Space $CcMP$; and if the Asymptote AS be not perpendicular to As , and so the Ordinate PM parallel to it, not perpendicular to the Absciss AP ; the Measure of the Ratio of AP to PM , with the Parallelogram ACc as a Module, will be still the Area of the Space $CcMP$.

This is demonstrated synthetically *Page 12.*
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EXAM-

EXAMPLE IX.

FIG. 9. 26. **T**O find the Area of the Space *ACMP* contained under the Part *AP*(*x*) of an infinite right Line perpendicular to the Axis of the Hyperbola, the Line *AC* (*a*) being the Continuation of the same Axis, and any Line or Ordinate *PM*(*y*) parallel to *AC*.

Here $aa + xx = yy$; and so $y = \sqrt{aa + xx}$. Consequently $\dot{x}y = \dot{x}\sqrt{aa + xx}$. The Fluent of which (by Case 3. Sect. 1.) will be $= ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$, &c. = Area *ACMP* sought. Which gives the Quadrature of the Segment *DCM* of the Hyperbola, by subtracting the same from the Rectangle *ADMP* ($\dot{x}y$); and if *a* be = 1, then will the Series be $x + \frac{x^3}{6} - \frac{x^5}{40} + \frac{x^7}{112} - \frac{5x^9}{1152a^7}$, &c.

Otherwise by the Measure of a Ratio or Angle.

The Fluxion $\dot{x}\sqrt{aa + xx}$ comes under the fourth Form of the Tables of Cotes. For if $z = x$, $n = 2$, $\theta = 0$, $d = 1$, $e = aa$, $f = 1$, then will $d\dot{z}z^{\theta n + \frac{1}{2}n - 1}\sqrt{e + fz^n}$ be $= \dot{x}\sqrt{aa + xx}$. The Fluent of which is $\frac{z^n}{n}dP + \frac{e}{nf}dR \left| \frac{R+T}{S} \right|$; and making $P \left(= \sqrt{\frac{e+fz^n}{z^n}} \right) = \frac{1}{x}\sqrt{aa + xx}$, $R (= \sqrt{f}) = 1$, $T \left(= \sqrt{\frac{e+fz^n}{z^n}} \right) = \frac{1}{x}\sqrt{aa + xx}$, $S \left(= \sqrt{\frac{e}{z^n}} \right) = \frac{a}{x}$. The said Fluent will become

$$\text{come } \frac{x}{2} \sqrt{aa+xx} + \frac{aa}{2} \left| \frac{x + \sqrt{aa+xx}}{a} \right| = \frac{1}{2} AP \\ \times PC + \frac{1}{2} AC^2 \left| \frac{AP+PC}{AC} \right| = \text{Area } ACPM;$$

that is, $\frac{1}{2} AP \times PC$ plus the Measure of the Ratio of $AP+PC$, and AC to $\frac{1}{2} AC$, as a Module will be the said Area.

The Quadrature of the Hyperbolick Space AMP may be had thus, as laid down by Mr. Cotes in *Harmonia Mensurarum*.

Make the Semi-conjugate Diameter $CB=b$, FIG. 10.

the Semi-transverse $CA=a$, $CP=x$, $PM=y$;

then from the Nature of the Curve we have $\frac{b}{a} \sqrt{aa-xx} = y$; and so $\frac{b}{a} x \sqrt{aa-xx} = \text{Fluxion of the Space } AMP$.

Now making $d = \frac{b}{a}$, $z=x$, $\theta=0$, $n=2$, $e=-aa$, $f=1$; the Fluxion of the 4th Form in the Tables of Mr. Cotes, viz. $d^2 z^{\theta n + \frac{1}{2}n - 1} \sqrt{e+fz^n}$ will become $\frac{b}{a} x \sqrt{xx-aa}$. And the

Fluent of it against $\theta=0$, is $\frac{z^n}{n} dP + \frac{e}{nf} dR$

$\left| \frac{R+T}{S} \right|$. And making $P (= \sqrt{\frac{e+fz^n}{z^n}}) = \frac{1}{x}$

$\sqrt{xx-aa}$, $R (=f) = 1$, $T (= \sqrt{\frac{e+fz^n}{z^n}}) = \frac{1}{x}$

$\sqrt{xx-aa}$, $S (= \sqrt{\frac{e}{z^n}}) = \frac{a}{x}$; the same will be-

come $\frac{bx}{2a} \sqrt{xx-aa} - \frac{ab}{2} \left| \frac{x + \sqrt{xx-aa}}{a} \right| =$

$\frac{xy}{2} - \frac{ab}{2} \left| \frac{a}{x - \frac{ay}{b}} \right|$, by substituting y for its Equal

$\frac{b}{a}\sqrt{xx-aa}$. And *Art. 11.* where the Ratio of $x + \sqrt{xx-aa}$ to a , is shewn to be equal to the Ratio of a to $x - \sqrt{xx-aa} = x - \frac{ay}{b}$.

Therefore $\frac{xy}{2} - \frac{ab}{2} \bigg| \frac{a}{x - \frac{ay}{b}}$ is the Fluent of the

Fluxion $\frac{b}{a} \dot{x} \sqrt{xx-aa}$. Which may be thus constructed.

Assume $CF : CA(a) :: PM(y) : CB(b)$; that is; make $CF = \frac{ay}{b}$. And assume $CG : CA(a)$

$:: CB(b) : PM(y)$; that is, make $CG = \frac{ab}{y}$.

Then if PH be taken equal to the Measure of the Ratio between $CA(a)$ and $FP (= x - \frac{ay}{b})$

to the Module $\frac{ab}{y}$; that is, if PH be taken

$= \frac{ab}{y} \bigg| \frac{a}{x - \frac{ay}{b}}$: And you draw the right Line

MH ; the right-lined Triangle HMP is = Space AMP . For it is $= \frac{y}{2} \times x - \frac{ab}{y}$

$\bigg| \frac{a}{x - \frac{ay}{b}} = \frac{yx}{2} - \frac{ab}{2} \bigg| \frac{a}{x - \frac{ay}{b}}$. And to find the

Quadrature of the external Hyperbolick Space

FIG. 11. $CAMP$ after the same manner, we may proceed thus.

Let AC, CB be Semi-conjugate Diameters, $AC = a, CB = b, PM = x, CP = y$; then

$\frac{b}{a}$

$\frac{b}{a}\sqrt{xx-aa}$ is $=y$, from the Nature of the Curve.

This Equation thrown into Fluxions, and there will arise $\frac{bbxx}{aay} = \dot{y} = \frac{bbxx}{ab\sqrt{xx-aa}} = \frac{bxx}{a\sqrt{xx-aa}}$.

But $\dot{y} \times x$ is $= \frac{bxxx}{a\sqrt{xx-aa}} =$ Fluxion of the Hyperbolick Space $CAMP$.

Now making $d = \frac{b}{a}$, $z = x$, $n = 2$, $\theta = 1$, $e = -aa$, $f = 1$; the Fluxion $\frac{d \dot{z} z^{\theta n + \frac{1}{2}n - 1}}{\sqrt{e + fz^n}}$ of the sixth Form in the Tables will become $\frac{bx^2 \dot{x}}{a\sqrt{xx-aa}}$. And the Fluent of it against $\theta = 1$,

will be $\frac{z^n}{nf} dP - \frac{e}{nff} dR \left| \frac{R+T}{S} \right.$. Whence

making $P (= \sqrt{\frac{e + fz^n}{z^n}}) = \frac{1}{x} \sqrt{xx-aa}$, $R (= \sqrt{f})$

$= 1$, $T (= \sqrt{\frac{e + fz^n}{z^n}}) = \frac{1}{x} \sqrt{xx-aa}$, $S (= \sqrt{\frac{e}{z^n}})$

$= \frac{a}{x}$; the said Fluent will become $\frac{bx}{2a} \sqrt{xx-aa}$

$= \frac{ab}{2} \left| \frac{x + \sqrt{xx-aa}}{a} \right| = \frac{xy}{2} + \frac{ab}{2} \left| \frac{a}{x - \frac{ay}{b}} \right|$, (by

substituting y for $\frac{b}{a} \sqrt{xx-aa}$, and Art. 11.) $=$

Fluent of the Fluxion $\frac{bx^2 \dot{x}}{a\sqrt{xx-aa}}$. The Construction of which may be thus.

Assume $CF:CB(b)::PM(x):AC(a)$; that is, make $CF = \frac{bx}{a}$; and $CG:CB(b)::AC$

A a 2

(a)

(a): $PM(x)$; that is, make $CG = \frac{ba}{x}$. Then if CH be taken equal to the Measure of the Ratio between $BC(b)$ and $PF\left(\frac{bx}{a} - y\right)$.

(Which is equal to the Ratio of a to $x - \frac{ay}{b}$)

to the Module $CG\left(\frac{ba}{x}\right)$: the said CH will be $= \frac{ba}{x} \left| \frac{a}{x - \frac{ay}{b}} \right|$. And you draw the right Line

MH , the right-lined Triangle is equal to the Space $CAMP$. For it is $= \frac{x}{2} \times y + \frac{ba}{x}$

$\left| \frac{a}{x - \frac{ay}{b}} = \frac{xy}{2} + \frac{ba}{2} \left| \frac{a}{x - \frac{ay}{b}} \right| \right|$, the Fluent of the given Fluxion.

EXAMPLE X.

27. **T**O square the Circle; or, which is the same thing, to find the Area of any Semi-segment APM thereof.

Make $AB=1$, $AP=x$, $PM=y$. Then from the Nature of the Circle $AP \times PB = \overline{PM}^2$; that is, $x - xx = yy$; and so $y = \sqrt{x - xx}$. Consequently $\dot{x}y = \sqrt{x - xx} = PM \times Pp$ is the Fluxion of the Area AMP . The Fluent of which (by Case 3. Sect. 1.) will be $\frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{7}x^{\frac{7}{2}} - \frac{2}{9}x^{\frac{9}{2}}$, &c. = Area AMP , or $x^{\frac{3}{2}}$ into $\frac{2}{3}x - \frac{2}{5}x^2 - \frac{2}{7}x^3 - \frac{2}{9}x^4$, &c.

Otherwise.

If the Radius CM be $= a$, and CP be supposed $= x$, $PM = y$; then from the Nature of the Circle $\overline{CM}^2 = \overline{CP}^2 + \overline{PM}^2$; that is, $aa = xx + yy$, $yy = aa - xx$, and $y = \sqrt{aa - xx}$. Therefore $\dot{x}y = \dot{x}\sqrt{aa - xx}$ is the Fluxion of the Indeterminate Space $PMDC$, and the Fluent of this, viz. $ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$, $\mathcal{E}c.$ = Area $PMDC$. Now if a be $= 1 = x$, then this Series will become $1 - \frac{1}{8} - \frac{1}{40} - \frac{1}{112} - \frac{1}{1152}$, $\mathcal{E}c.$ = equal to the Area of the Quadrant ADC . And four times this will be the Quadrature of the whole Circle $ADBA$; or if the Diameter be 1, the said Series will express the Area of the whole Circle.

Otherwise:

Let AE the Tangent of $\frac{1}{2}$ the Arch AM FIG. 13. be $= x$, the Radius $AC = 1$. Let AB be the Tangent of the Arch AM . Draw the Secants CE , CB , and the Sine MP of the Arch AM . Let pm be infinitely near PM , and from the Centre C draw the Secant Cb thro' the Point m ; also from the Point M draw Mt perpendicular to pm , and from the Point B draw Bs perpendicular to Cb .

Now we propose here to find the Area of the indefinite Sector ACM ; the Fluxion of which, being the little Sector MCm , must first be found thus:

First,

First, the Tangent AB of the Arch BM will be $= \frac{2x}{1-xx}$. Now because the Angle ACB is bisected by the right Line CE , therefore $AE(x):AC(1)::EB\left(\frac{x+x^3}{1-xx}\right):CB=\frac{1+x^2}{1-x^2}$. Also because of the similar Triangles $ACB, PCM, CB\left(\frac{1+x^2}{1-x^2}\right)::AB\left(\frac{2x}{1-x^2}\right)::AC(1):PM=\frac{2x}{1+x^2}$, and $CB\left(\frac{1+x^2}{1-x^2}\right)::AC(1)::CM(1):CP=\frac{1-x^2}{1+x^2}$. Whence $AP\frac{2x^2}{1+x^2}$; the Fluxion of which will be $\frac{4x\dot{x}}{1+xx} = Pp$ or Mt .

Again, the little Triangle Mmt right-angled at t , will be similar to the right-angled Triangle CMP , the Angle tMm being = Angle PMC , and the Angle tmM = Angle PCM , as is easy to prove. Therefore $MP\left(\frac{2x}{1+xx}\right):AC(1)::Mt\left(\frac{4x\dot{x}}{1+xx^2}\right):Mm=\frac{2\dot{x}}{1+xx}$. And so $\frac{1}{2}MC(1)\times Mm=\frac{\dot{x}}{1+xx}$ = Area of the little Sector MCm , being the Fluxion of the Sector AMC . The Fluent of which will be $x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\frac{1}{9}x^9$, &c. = Area of the Indeterminate Sector AMC . And when the Tangent $AE(x)$ of half the Arch AM becomes $=1$ = Radius, then the Sector ACM will become a Quadrant; and the Series aforesaid, expressing the Area of the same, will be $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}$, &c.

ℰc. and if the Diameter of the Circle be = 1, the whole Area will be expressed by that Series.

The said Series may be found shorter thus: If AB be = x , then will $Bb = \dot{x}$, and $CB = \sqrt{1 + xx}$. Now the little Triangle BSb right-angled at S , is similar to the Triangle ABC , the Angle ABC differing from the Angle b only an infinitely small Quantity, and so they may be taken for Equals; therefore $CB(\sqrt{1 + xx})$

$$: AC(1) :: Bb(\dot{x}) : Bs = \frac{\dot{x}}{\sqrt{1 + xx}}. \text{ Moreo-}$$

ver, since Bs is infinitely small, CB and Cs differ from one another only by an infinitely small Quantity: therefore $CB(\sqrt{1 + xx}) : Bs$

$$\left(\frac{\dot{x}}{\sqrt{1 + xx}} \right) :: MC(1) : Mm = \frac{\dot{x}}{1 + xx}. \text{ Whence}$$

$$\text{the little Sector } MCm = \frac{\dot{x}}{2 + 2xx} = \text{Fluxion}$$

of the Area of the Sector AMC . The Flu-ent of which will be $\frac{1}{2}x - \frac{1}{8}x^3 + \frac{1}{16}x^5 - \frac{1}{128}x^7 + \frac{1}{128}x^9$, ℰc. = Area of the said Sector: so that when the Sector ACM is an eighth Part of the Circle, viz. when the Tangent $AB(x)$ = Radius AC is = 1; then the aforefaid Series will become $\frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{1}{128} + \frac{1}{128}$, ℰc. which doubled will be $1 - \frac{1}{4} + \frac{1}{8}$, ℰc. = Area of the Quadrant as before.

EXAMPLE XI.

28. **T**O square the Elliptic Space: or, which is the same thing, to find the Area of any Indeterminate Elliptic Segment $ACMP$ contain'd under the Semi-conjugate Diameter AC , the Or-
dinate

FIG. 14

dinate PM, Part of the Absciss AP, and CM Part of the Curve of the Ellipsis.

Call AC, a , AB or AD, b , AP, x , PM, y ; then from the Nature of the Ellipsis $\overline{MP}^2 = \frac{\overline{AD}^2 \times \overline{AB}^2 - \overline{AP}^2}{\overline{AC}^2}$, that is, $yy = \frac{bb \times aa - xx}{aa}$;

and so $y = \frac{b}{a} \sqrt{aa - xx}$. Consequently $\frac{b}{a} x$

$\sqrt{aa - xx}$ will be the Fluxion of the Space $ACPM$, and the Fluent of this will be $bx - \frac{bx^3}{6a^2} - \frac{bx^5}{40a^4} - \frac{bx^7}{112a^6}$, &c. = Area $ACPM$.

Now if you put a for x in this Series, it will become $ab - \frac{1}{6}ab - \frac{1}{40}ab - \frac{1}{112}ab$, &c. = Area of the Quadrant ACD of the Ellipsis. And if a be = Axis BD , then this last Series will express the Area of the whole Ellipsis. And if $\sqrt{ab}$ be = 1, then the Area of the Ellipsis will be $1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{1}{1152}$, &c. and therefore an Ellipsis is equal to a Circle, whose Diameter is a mean Proportional between the Transverse and conjugate Diameters of the Ellipsis. And so an Ellipsis is to a Circle of the same Diameter with the transverse Axis, as ab to a^2 , or b to a , viz. as the conjugate Axis is to the transverse Axis.

Otherwise.

To find the Area of any Sector CMA of the Ellipsis.

FIG. 15.

24. Let CB be the Semi-conjugate, and CA the Semi-transverse Diameter, MP a Semi-Ordinate. Now draw mp infinitely near MP ,

MP, join the Points *C* and *M*, and *m* by the right Lines *CM*, *Cm*; and from *m* draw the short right Line *mH* perpendicular to *MP*, cutting *MP* in *I*, and *CM* in *H*; as also the little Line *mK* perpendicular to *CM* continued out.

This done, let $AC=a$, $BC=1$, $AP=x$, $PM(y)$, $CM=u$, and $CP=z$. Now the first thing to be found must be the Area of the small Triangle *CMm*. Thus: Because the Triangles *CPM*, *HIM* are similar; therefore

$PM(y):CP(z)::MI(\dot{y}):IH=\frac{z\dot{y}}{y}$. Whence (since $Pp=Im=\dot{x}=-\dot{z}$) $Hm=\frac{z\dot{y}}{y}-\dot{z}$. A-

gain; because of the similar Triangles *CMP*, *HKM*. $CM(u):PM(y)::Hm\left(\frac{z\dot{y}}{y}-\dot{z}\right):$

$mK=\frac{z\dot{y}-y\dot{z}}{u}$. Which drawn into half the

Base *CM(u)*; and then the Area of the fluxionary Triangle *CMm* will be $=\frac{z\dot{y}-y\dot{z}}{2}$; and

throwing the Equation of the Curve $\frac{aa-zz}{aa}=\dot{y}$ into Fluxions we have $-\frac{\dot{z}}{aay}=\dot{y}$.

Which substitute in $\frac{z\dot{y}-y\dot{z}}{2}$ for $\dot{y}$, and there

arises $\frac{-zz\dot{z}}{2aay}-\frac{y\dot{z}}{2}=\frac{-zz\dot{z}-aayy\dot{z}}{2aay}$. In which

substituting $aa-zz$ for its Equal $aayy$, and

it will be $\frac{-zz\dot{z}-aa\dot{z}+zz\dot{z}}{2aay}=\frac{-aa\dot{z}}{2ay}=\frac{a\dot{x}}{2ay}$;

since $\dot{x}=-\dot{z}$. And again substituting $\sqrt{2ax-xx}$ for *ay*; and the Triangle *CMm* will

FIG. 16. $be = \frac{az}{2\sqrt{2ax-xx}} = \text{Fluxion of the Sector } ACM$
 of the Ellipsis. And if you make $\sqrt{2ax-xx} = \frac{n}{x}$; then will x be found $= \frac{2ann}{1+nn}$, and $\dot{x} = \frac{4ann\dot{n}}{1+nn}$; and by due Substitution $\frac{ax}{2\sqrt{2ax-xx}} = \frac{an}{1+nn}$. The Fluent of which is $an - \frac{an^3}{3} + \frac{an^5}{5} - \frac{an^7}{7}$, $\mathcal{E}c.$ = Sector of the Ellipsis.

And after the same manner you will find the Sector of the Hyperbola (Fig. 16.) to be $an + \frac{an^3}{3} + \frac{an^5}{5} + \frac{an^7}{7}$, $\mathcal{E}c.$ and the Sector of the Circle will be $= n - \frac{n^3}{3} + \frac{n^5}{5} - \frac{n^7}{7}$, $\mathcal{E}c.$ and this becomes $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$, $\mathcal{E}c.$ by making $n=1$.

This may be done something shorter, thus:

FIG. 17, 18. Draw the Tangent AD , and continue out CM to cut the same in D . Draw Cd infinitely near CD , and from C describe the small Arches Mn, De . Make $CB=b$, $CA=a$, $AD=x$, $CD=y$, $CP=z$; because the Triangles CAD, Ded are similar, the Angle A differing from the Angle D only by an infinitely small Angle, which may be rejected, and the Angles at e , and A right ones. Therefore $CD(y):AC(a)::Dd(\dot{x}):De = \frac{ax}{y}$. And because of the similar Triangles CPM, CAD , $AC(a):CD(y)::CP(z):CM\left(\frac{yz}{a}\right)$. Likewise since the Sectors CDe, CMn are similar, $CD(y):De$

$De\left(\frac{ax}{y}\right)::CM\left(\frac{yz}{a}\right):Mn=\frac{zx}{y}$. Now

$\frac{1}{2}CM \times Mn$, that is, $\frac{zx}{2y} \times \frac{yz}{a} = \frac{zzx}{2a}$ is = A-

rea of the Triangle CMm , being the Fluxion of the Sector CAM . Again, from the Nature of the Curve, and the Similarity of the Triangles CPM , CAD , in the Ellipsis we have $PM\left(\frac{b}{a}\sqrt{aa-zz}\right):CP(z)::AD(x):$

$AC(a)$; and in the Hyperb. $PM\left(\frac{b}{a}\sqrt{-aa+zz}\right)$

$:CP(z)::AD(x):AC(a)$. And multiplying the Means and Extremes, we have $zx = b\sqrt{aa+zz}$ in the Ellipsis, and $zx = b\sqrt{-aa+zz}$

in the Hyperbola. Consequently $zz = \frac{aabb}{bb+xx}$ in the Ellipsis, and $zz = \frac{aabb}{bb-xx}$ in the Hyperbola.

Which being substituted in $\frac{zzx}{2a}$, we

have $\frac{abbx}{2bb+2xx}$ in the Ellipsis, and $\frac{abbx}{2bb-2xx}$

in the Hyperbola, = Fluxion of the Area of the Sector CAM , the Fluent of which will

be $\frac{ax}{2} - \frac{ax^3}{6bb} + \frac{ax^5}{10b^4} - \frac{ax^7}{14b^6}$, &c. in the Ellip-

fis, and $\frac{ax}{2} - \frac{ax^3}{6bb} - \frac{ax^5}{10b^4} - \frac{ax^7}{14b^6}$, &c. in the

Hyperbola = Area of the Sector CAM ; and making $CB(b) = 1$, the said Fluent will be

$\frac{1}{2}ax - \frac{1}{6}ax^3 + \frac{1}{10}ax^5 - \frac{1}{14}ax^7$, &c. in the El-

lipsis, and $\frac{1}{2}ax - \frac{1}{6}ax^3 - \frac{1}{10}ax^5 - \frac{1}{14}ax^7$, &c. in the Hyperbola.

And in the Ellipsis, if x be = b , the Area of the Sector CAM will be $\frac{1}{2}ab - \frac{1}{6}ab + \frac{1}{10}ab - \frac{1}{14}ab$, &c. and when x

$=a$, the Area of the Sector CBM will be also $\frac{1}{2}ab - \frac{1}{8}ab + \frac{1}{16}ab - \frac{1}{24}ab$, &c. the Sum of which, viz. $ab - \frac{1}{8}ab + \frac{1}{16}ab - \frac{1}{24}ab$, &c. will be the Area of the Quadrant of the Ellipsis ABC .

COROL.

HENCE if $a=b$, viz. when the Ellipsis is a Circle; the Area of the Quadrant of a Circle, whose Radius is a , will be $aa - \frac{1}{8}aa + \frac{1}{16}aa - \frac{1}{24}aa$, &c. and if a be $=1$, the Area of the Quadrant will be $1 - \frac{1}{8} + \frac{1}{16} - \frac{1}{24}$, &c.

The Fluent of $\frac{abbx}{2bb-2xx}$, the Fluxion of the Sector of the Hyperbola, may be found in the Measure of a Ratio; for it may be referr'd to the second Form in the Tables of Mr. Cotes; since making $z=x$, $d=abb$, $\theta=0$, $n=2$, $e=2bb$, $f=-2$, the Fluxion in the second Form $\frac{dz z^{\theta n + \frac{1}{2}n - 1}}{e + fz^n} = \frac{abbx}{2bb-2xx}$, and the Fluent of it against $\theta=0$, is $\frac{2}{ne} dR \left| \frac{R+T}{S} \right.$. Now $R (= \sqrt{\frac{e}{f}}) = b$, $T (= z^{\frac{1}{2}n}) = x$, and $S (= \sqrt{\frac{e+fz^n}{f}}) = \sqrt{bb-xx}$. Whence $\frac{2}{ne} dR \left| \frac{R+T}{S} \right. = \frac{1}{2}ab \left| \frac{b+x}{\sqrt{bb-xx}} \right. = \frac{1}{2}AC \times CB \left| \frac{CB+AD}{\sqrt{CB^2-AD^2}} \right.$; that is, the Area of the Hyperbolical Sector CAM is equal to the Measure of the Ratio of $CB+AD$ to $\sqrt{CB^2-AD^2}$, the Triangle ACB being the Module.

EXAM.

EXAMPLE XII.

30. **T**O square the Space AMp , called the Figure of the Tangents.

The Nature of this Figure is, that any Absciss AP is equal to any Arch (AP) of a Circle, and the correspondent Ordinate PM at right Angles to it, is equal to the correspondent Tangent AT of the Arch. FIG. 19, 20.

Draw the Secant CT , and the Secant Ct infinitely near to it; and from C with the Distance CT describe the small Arch Tc . Make the Radius $AC = a$, the Arch $AP (= \text{Absciss } AP) = x$, and the correspondent Tangent $AT (= \text{correspondent Ordinate } pM) = y$. Now because the Triangles ATC , Ttc , are similar, the Angles at T , t , differing from each other only by an infinitely small Angle, and the Angles at A and C right Angles; therefore $TC (\sqrt{aa + yy}) : AC (a) :: Tt (\dot{y}) : Tc =$

$\frac{a\dot{y}}{\sqrt{aa + yy}}$. Again, because of the similar Secantors CPp , CTc , therefore $CT (\sqrt{aa + yy}) : Tc$

$\left(\frac{a\dot{y}}{\sqrt{aa + yy}} \right) :: CP (a) : Pp (= \dot{x})$. And multiplying the Means and Extremes, we get $\dot{x}$

$\sqrt{aa + yy} = \frac{aay}{\sqrt{aa + yy}}$; and dividing by $\sqrt{aa + yy}$,

there will be had $\dot{x} = \frac{aay}{aa + yy} = \text{Fluxion of the Absciss } Ap$; and multiplying by y , there

will arise $\dot{x}y = \frac{aay\dot{y}}{aa + yy} = \text{Fluxion of the Space}$

I

AMp ;

AMp ; and the Fluent of it will be $yy - \frac{y^4}{4aa} + \frac{y^6}{6a^4} - \frac{y^8}{8a^6} +, \&c. = \text{Area sought.}$

Otherwise by the Measure of a Ratio.

This Fluxion $\frac{aay\dot{y}}{aa+yy}$ may be compared with the first Form in Mr. Cotes's Tables. For if $z=y$, $n=2$, $\theta=1$, $d=aa$, $e=aa$, $f=1$;

the first Form $\frac{dz z^{\theta n-1}}{e+fz^n}$ will be $= \frac{aay\dot{y}}{aa+yy}$.

And against $\theta (=1)$, the Fluent will be $\frac{d}{nf} \left| \frac{e+fz^n}{e} \right| = \frac{1}{2}aa \left| \frac{aa+yy}{aa} \right| = \frac{1}{2} \overline{AC}^2 \left| \frac{\overline{CT}^2}{\overline{AC}^2} \right|$, which

is equal to $\overline{AC}^2 \left| \frac{CT}{AC} \right|$. For putting $2l$ for the

Logarithm of the Ratio $\frac{\overline{CT}^2}{\overline{AC}^2}$, and m for the

Module of the Logarithms; then will $\frac{1}{2} \overline{AC}^2$

* Art. 11. $\frac{\overline{CT}^2}{\overline{AC}^2}$ be $*$ $= \frac{1}{2} \overline{AC}^2 \times \frac{2b}{m} = \overline{AC}^2 \times \frac{l}{m}$. But from the Nature of the Logarithms l is equal to the Logarithm of the Ratio $\frac{CT}{AC}$; therefore $\overline{AC}^2$

* Art. 11. $\times \frac{l}{m}$ is $*$ $= \overline{AC}^2 \left| \frac{CT}{AC} \right| = \frac{1}{2} \overline{AC}^2 \left| \frac{\overline{CT}^2}{\overline{AC}^2} \right|$; therefore

the Area ApM is equal to the Measure of the Ratio between the Secant CT of the Arch $AP=Ap$, and the Radius AC , having the Square of the Radius for a Module.

EXAMPLE XIII.

31. **TO** square the Space $ABpM$, called the Figure of the Secants.

Here the Absciss Ap is equal to the Arch FIG. 21.
 AP , as in the last Example; but the corresponding Ordinate pM is = Secant CT , and the Ordinate AB = Radius AC , every thing else being as in the last Example; only let CT be = y . Then from the Similarity of the Triangles ACT , Ttc , we have $AT(\sqrt{yy - aa})$

$:: AC(a) : tc(\dot{y}) : Tc = \frac{a\dot{y}}{\sqrt{yy - aa}}$; and because

of the similar Sectors CTc , CPp , therefore

$CT(y) : Tc\left(\frac{a\dot{y}}{\sqrt{yy - aa}}\right) :: CP(a) : Pp(\dot{x})$;

and multiplying the Means and Extremes, there

arises $y\dot{x} = \frac{aa\dot{y}}{\sqrt{yy - aa}}$ = Fluxion of the Space

$ABMp$, and the Fluent thereof will be very easily found in the Measure of a Ratio, it coming under the sixth Form of Mr. Cotes's Tables. For making $z=y$, $n=2$, $\theta=0$, $d=aa$,

$e=-aa$, $f=1$, we shall have $\frac{dz z^{\theta n + \frac{1}{2}n - 1}}{\sqrt{e + fz^n}} =$

$\frac{aa\dot{y}}{\sqrt{-aa + yy}}$. Also in this Form $R(=\sqrt{f})=1$,

$T\left(=\frac{fz^{\frac{1}{2}n}}{\sqrt{e + fz^n}}\right) = \frac{y}{\sqrt{-aa + yy}}$, and

$S\left(=\sqrt{\frac{-ef}{e + fz^n}}\right) = \frac{a}{\sqrt{-aa + yy}}$. Whence

the

the Fluent (θ being $= 0$) $\frac{uf}{2} dR \left| \frac{R+T}{S} \right|$ will

$$\text{be } aa \left| \frac{y + \sqrt{-aa + yy}}{a} \right| = \overline{AC}^2 \left| \frac{CT + AT}{AC} \right|.$$

Consequently the Area of the Space $ABMp$ will be equal to the Measure of the Ratio between the Sum of the Secant and Tangent of the same Arch, and the Radius to the Square of the Radius for a Module.

COROL.

HENCE the Meridional Parts in *Mercator's* Chart may be computed for any given Latitude AP . It is well known*, that the Meridional Parts of any given Latitude AP , are to the Length of the Arch AP , as the Sum of the Secants in them to the Sum of as many Semidiameters; that is, as the Area of the Curve-lin'd Space $ABMp$ to the Rectangle $BA \times pA$; or as $\overline{AC}^2 \left| \frac{CT + AT}{AC} \right|$ to $AC \times AP$, or as $AC \left| \frac{CT + AT}{AC} \right|$ to AP . Whence the Meridional Parts are equal to $AC \left| \frac{CT + AT}{AC} \right|$

* Art. 14. or * $AC \left| \frac{AC}{CT - AT} \right|$

EXAMPLE XIV.

32. TO find the Area of the Space $CPQc$ contained under the Parts of the Conchoids CPD, cqd of Nicomedes, the Part Cc of the right Line AC drawn from the Pole A perpendicular to the common Asymptote BG , and the

* See *Philosophical Trans.* N^o. 176,

the Part QP of any right Line drawn from the said Pole A, and intercepted between the Curves.

Let $Ac = a$, $cB = CB = b$, $cC = 2b$, $AB = c$, $AG = y$, $BG = x$. Draw Ap infinitely near AP ; and from A , as a Centre, describe the little Concentrick Arches qs , mr , pn . FIG. 22.

Now the Nature of these Curves is, that BC or Bc is $= GP$ (pm) or GQ (mq). Because the Triangles AGB , rGm are similar, the Angle at G being common, and the Angles at r and B right ones (the Arch rm being taken for a straight Line) therefore AG (y): AB (c)

$:: GM$ ($\dot{x}$): $rm = \frac{c\dot{x}}{y}$. Again, since the Sectors Arm , Asq are similar, therefore $Am = AG$ (y): $rm \left(\frac{c\dot{x}}{y} \right) :: Aq = AQ = AG - AQ$

$(y - b):sq = \frac{cy\dot{x} - cb\dot{x}}{yy}$. And again, from the

Similarity of the Sectors Arm , Anp ; therefore AG (y): $mr \left(\frac{c\dot{x}}{y} \right) :: AP$ ($y + b$): $pn =$

$\frac{cy\dot{x} + cb\dot{x}}{yy}$. Consequently $pn + qs$ is $= \frac{2c\dot{x}}{y}$.

And so $\frac{1}{2}pn + qs \times PQ = \frac{2cb\dot{x}}{y}$ is $=$ Area of the little Space $sPpq =$ Space $QPpq =$ Fluxion of the Space $CPQc$. And substituting

$\sqrt{cc + xx}$ for y , the Fluxion will be $\frac{2cb\dot{x}}{\sqrt{cc + xx}}$.

Which comes under the sixth Form in the Tables of Mr. Cotes. For making $d = 2cb$, $z = x$, $\theta = 0$, $n = 2$, $e = cc$, $f = 1$, the said sixth Form

$\frac{dz z^{\theta n + \frac{1}{2}n - 1}}{\sqrt{e + fz^n}}$ will become $\frac{2cb\dot{x}}{\sqrt{cc + xx}}$. The Fluxion

cent of which standing against $\theta=0$, is $\frac{2}{nf} dR$
 $\left| \frac{R+T}{S} \right.$. And making $R (= \sqrt{f}) = 1$, $T (=$
 $\sqrt{\frac{e+fz^n}{z^n}}) = \frac{1}{x} \sqrt{cc+xx}$, and $S (= \sqrt{\frac{e}{z^n}}) =$
 $\frac{c}{x}$, the said Fluent will become $2bc \left| \frac{x+\sqrt{cc+xx}}{c} \right.$
 $=$ Fluent of the given Fluxion $=$ Area of the
 Space $CPQc$, $= Bc \times AB \left| \frac{BG+AG}{AB} \right.$. That
 is, the Area of the said Space is equal to a Re-
 ctangle under Bc , and the Measure of the Ra-
 tio between $BG+AG$, and AB to the Mo-
 dule AB .

FIG. 23. But to find the Area of the Space $CPGB$
 contained under the Part PC of only one of the
 Conchoides of Nicomedes, BE being the Asym-
 ptote, A the Pole, and ABC at right Angles to
 BE .

Call AB, a , BC, b , AG, y , AB, x . Draw
 Ap infinitely near to AP , and from A describe
 the small Arches mr, pn . Then by reasoning
 as before, we shall have $\frac{abbx}{2aa+2xx} + \frac{abx}{\sqrt{aa+xx}} =$

Area of the little Space $GPpm =$ Fluxion of
 the Space $CPGB$. The first Part of which
 Fluxion comes under the second Form in the
 Tables of Mr. Cotes: for making $d=abb$, $z=x$,
 $\theta=0$, $n=2$, $e=2aa$, $f=2$, the Fluxion of the
 second Form, viz. $\frac{dz z^{\theta n + \frac{1}{2}n - 1}}{e + fz^n}$ will become

$\frac{abbx}{2aa+2xx}$; the Fluent of which is $\frac{2}{ne} dR$
 $\left| \frac{R+T}{S} \right.$.

$\left| \frac{R+T}{S} \right.$ In which substituting for $R (= \sqrt{\frac{-e}{f}})$
 $\sqrt{-aa}$, for $T (= z^{\frac{1}{2}n}) x$, and for $S (= \sqrt{\frac{e+fz}{f}})$

$\sqrt{aa+xx}$, and we shall have $\frac{1}{2}bb \left| \frac{a+x}{\sqrt{aa+xx}} \right.$

for the Fluent of the first Part $\frac{abbx}{2aa+2xx}$.

Again, the latter Part $\frac{abx}{\sqrt{aa+xx}}$ of the Fluxion may be compared with the sixth Form in the Tables of Mr. Cotes by making $d=ab$, $z=x$, $\theta=0$, $n=2$, $e=aa$, $f=1$; and the Fluent thereof against $\theta=0$, viz. $\frac{2}{nf} dR \left| \frac{R+T}{S} \right.$,

by making $R=1$, $T=\frac{1}{x}\sqrt{aa+xx}$, and $S=\frac{a}{x}$,

will be $ab \left| \frac{x+\sqrt{aa+xx}}{a} \right.$. Therefore the Sum

of these Fluents $\frac{1}{2}bb \left| \frac{a+x}{\sqrt{aa+xx}} \right.$ + ab

$\left| \frac{x+\sqrt{aa+xx}}{a} \right.$ will be the Fluent of the whole

Fluxion. Consisting likewise of two Parts, the former of which is the Measure of an Angle, because R is $=\sqrt{-aa}$; and the latter of a Ratio, because R is $=\sqrt{1}=1$.

The Construction of which Fluent may be thus: Draw CF perpendicular to AC , assume BH equal to the Measure of the Ratio between $BG+AG$, and AB to AB taken as a

Module, that is, make $BH=a \left| \frac{x+\sqrt{aa+xx}}{a} \right.$

FIG. 24.

Moreover, assume BI equal to the Measure of the Angle GAB , to the same Module AB , that is, $= a \left| \frac{a+x}{\sqrt{aa+xx}} \right|$. The Radius, Tan-

gent, and Secant of which being as a , x , and $\sqrt{aa+xx}$, or AB , BG , AG . This done, draw AIE and HD parallel to the same; then will the Trapezium $BHDC$ be equal to the whole Fluent; and consequently equal to the Space $CPGB$. For since the Triangles ABH , ACD are similar, therefore $AB (a) : BI$

$$\left(a \left| \frac{a+x}{\sqrt{aa+xx}} \right| \right) :: AC (a+b) : CE = a \left| \frac{a+x}{\sqrt{aa+xx}} \right| + b \left| \frac{a+x}{\sqrt{aa+xx}} \right|. \text{ And } \frac{CE + IB}{2}$$

$$\times BC = \frac{ab}{2} \left| \frac{a+x}{\sqrt{aa+xx}} \right| + \frac{bb}{2} \left| \frac{a+x}{\sqrt{aa+xx}} \right| =$$

Trapezium $BIEC$. Again, $HI (= BH - BI)$ is $= a \left| \frac{x + \sqrt{aa+xx}}{a} \right| - a \left| \frac{a+x}{\sqrt{aa+xx}} \right|$;

which multiplied by $BC (b)$, and the Product

$$ab \left| \frac{x + \sqrt{aa+xx}}{a} \right| - ab \left| \frac{a+x}{\sqrt{aa+xx}} \right| (= \text{Parallelogram } HDEI)$$

added to the Trapezium $BIEC$

$$\text{will be } \frac{ab}{2} \left| \frac{a+x}{\sqrt{aa+xx}} \right| + \frac{bb}{2} \left| \frac{a+x}{\sqrt{aa+xx}} \right| - \frac{ab}{2}$$

$$\left| \frac{a+x}{\sqrt{aa+xx}} \right| + ab \left| \frac{x + \sqrt{aa+xx}}{a} \right| = \frac{bb}{2} \left| \frac{a+x}{\sqrt{aa+xx}} \right|$$

$$+ ab \left| \frac{x + \sqrt{aa+xx}}{a} \right| = \text{Fluent to be constructed,} = \text{Trapezium } BHDC.$$

If

If the Conchoid CPD be of such a Nature, that $AB \times BC$ (ab) be $= AG \times GP$ (yb);

by reasoning as above, we shall have $\frac{aaby^{-1}y}{\sqrt{yy-aa}}$

+ $\frac{a^3bb^{-3}y}{2\sqrt{yy-aa}}$ for the Area of the little Space

$GPpm$ = Fluxion of the Space $GPCB$; and the Fluents of each Part of this Fluxion may be had from those of the Fluxion of the fifth Form in the Tables of Mr. Cotes. For making $d=aab$, $z=y$, $\theta=0$, $n=2$, $e=-aa$, $f=1$, the

Fluxion of the Form, viz. $\frac{dz z^{\theta n-1}}{\sqrt{e+fz^n}}$ will

become $\frac{aaby^{-1}y}{\sqrt{yy-aa}}$, the first Part of the

Fluxion. And making $d=\frac{a^3bb}{2}$, $z=y$, $\theta=-1$,

$n=2$, $e=-aa$, $f=1$, the said Fluxion in the

Form will become $\frac{a^3bby^{-3}y}{2\sqrt{yy-aa}}$, the latter Part

of the Fluxion; and the Fluent in the former

Case corresponding to $\theta=0$, is $\frac{-2}{ne} dR \left| \frac{R+T}{S} \right|$;

and in the latter, the Fluent against $\theta=-1$,

is $\frac{-1}{nez^n} dP + \frac{f}{nee} dR \left| \frac{R+T}{S} \right|$. Consequently

writing for P ($=\sqrt{e+fz^n}$) $\sqrt{yy-aa}$, for R

($=\sqrt{e}$) $\sqrt{-aa}$, for T ($=\sqrt{e+fz^n}$) $\sqrt{yy-aa}$,

and for S ($=\sqrt{fz^n}$) y ; the former Fluent will

become $ab \left| \frac{a+\sqrt{yy-aa}}{y} \right|$, and the latter $\frac{abb}{4yy}$

$\sqrt{yy-aa} + \frac{bb}{4} \left| \frac{a-\sqrt{yy-aa}}{y} \right|$; and the Sum of these

Fluents

Fluents $\frac{abb}{4yy} \sqrt{yy-aa} + ab + \frac{bb}{4} \left| \frac{a + \sqrt{yy-aa}}{y} \right|$
 is = Fluent of the Fluxion above. Which
 may be constructed thus:

FIG. 24. Make $AG(y) : \frac{1}{2}AB(\frac{1}{2}a) :: \frac{1}{2}BC(\frac{1}{2}b) : BN$
 $= \frac{ab}{4y}$. And $AG(y) : GB(\sqrt{yy-aa}) :: BN$
 $\left(\frac{ab}{4y}\right) : BM = \frac{ab}{4yy} \sqrt{yy-aa}$; then if to BM
 you add $MO =$ to the Measure of the Angle
 BAG (whose Radius, Tangent and Secant are
 as $AB(a)$, $BG(\sqrt{yy-aa})$, and $AG(y)$ to the
 Module $AB + \frac{BC}{4} \left(a + \frac{b}{4}\right)$, the Rectangle
 $CB \times BO$ will be equal to the Fluent
 $\frac{abb}{4yy} \sqrt{yy-aa} + ab + \frac{bb}{4} \left| \frac{a + \sqrt{yy-aa}}{y} \right| = \text{Area}$
 of the Space $GPCB$.

EXAMPLE XV.

33. **L**ET DPEBQD be half the Lune of Hip-
 pocrates, A being the Centre of the Arch
 DQB, and C the Centre of DPE; it is requi-
 red to find the Area of the Space QPEB, con-
 tained under BE, Part of a Line ACEB join-
 ing the Centres A, C; the Parts QB, PE of the
 Arches forming the Semi-Lune; and the Part
 QP of any right Line AQP drawn from the
 Centre A of the larger Arch of Formation, lying
 between the said Arches.

FIG. 25. Let DCF be at right Angles to ACE.
 Draw Agqp infinitely near AGQP, and from
 the Centre A describe the small Arches Gr, ps,
 and join the Points P and E.

This done, let $AC = a$, $CG = x$, $AG = y$.
 Because the Triangles AGC , rgG are similar;
 therefore $AG (y) : AC (a) :: Gg (\dot{x}) : Gr =$
 $\frac{a\dot{x}}{y}$. And since the Triangles AGC , AEP ,
 are likewise similar, the Angle A being com-
 mon to both, and the Angle APE in a Semi-
 circle equal to the right Angle ACG ; there-
 fore $AG (y) : AC (a) :: AE (2a) : AP =$
 $\frac{2aa}{y}$. Moreover, from the Similarity of the
 Sectors AGr , Asp , we have $AG (y) : Gr$
 $\left(\frac{a\dot{x}}{y}\right) :: AP \left(\frac{2aa}{y}\right) : sp = \frac{2a^2\dot{x}}{y^3}$. Which
 drawn into $\frac{1}{2} AP \left(\frac{aa}{y}\right)$, and the Product
 $\frac{2a^3\dot{x}}{y^4}$ will be = Area of the little Sector $Ap s$
 = little Triangle ApP . For these differ only
 by the Triangle pPs , which is infinitely less
 than $Ap s$. Again, because the Sectors AGr ,
 Asp are similar, we have $AG (y) : Gr \left(\frac{a\dot{x}}{y}\right)$
 $:: A\mathcal{Q} (a\sqrt{2}) : \mathcal{Q}g = \frac{aa\dot{x}\sqrt{2}}{yy}$. Which drawn
 into $\frac{1}{2} A \left(\frac{a}{2}\sqrt{2}\right)$, and the Product $\frac{a^3\dot{x}}{yy}$, will
 be equal to the Area of the little Sector or
 Triangle $A\mathcal{Q}g$; which taken from $\frac{2a^3\dot{x}}{y^4}$,
 the Area of the Triangle ApP , and the Re-
 mainder $\frac{2a^3\dot{x}}{y^4} - \frac{a^3\dot{x}}{yy}$ will be = Area of the Tra-
 pezium $\mathcal{Q}Ppq$. Which is the Fluxion of the
 lunar Space $EP\mathcal{Q}B$.

Again,

Again, because $yy = aa + xx$; therefore $xa = yy - aa$: and throwing this Equation into Fluxions, we get $x\dot{x} = y\dot{y}$, and $\dot{x} = \frac{y\dot{y}}{x} = \frac{y\dot{y}}{\sqrt{yy - aa}}$,

by putting for x its equal $\sqrt{yy - aa}$; then if this last Value be put for $\dot{x}$ in the Fluxion of the sought Lunar Space, the same will become

$$\frac{2a^2\dot{y}}{y^3\sqrt{yy - aa}} - \frac{a^2\dot{y}}{y\sqrt{yy - aa}} = \frac{2a^2y^{-3}\dot{y}}{\sqrt{yy - aa}} - \frac{a^2y^{-1}\dot{y}}{\sqrt{yy - aa}}.$$

Both Parts of which come under the fifth Form in the Tables of Mr. Cotes. For in the first Part, making $d = 2a^2$, $z = y$, $\theta = -1$, $n = 2$, $e = -aa$, $f = 1$, we have the Fluxion of the

$$\text{Form } \frac{dz z^{\theta n - 1}}{\sqrt{e + fz^n}} = \frac{2a^2 y^{-3} \dot{y}}{\sqrt{yy - aa}}.$$

In like manner, in the second Part, making $d = -a^2$, $z = y$, $\theta = 0$, $n = 2$, $e = -aa$, $f = 1$, we have

$$\frac{dz z^{\theta n - 1}}{\sqrt{e + fz^n}} = \frac{-a^2 y^{-1} \dot{y}}{\sqrt{yy - aa}}.$$

The Fluent against $\theta = -1$ is $\frac{-1}{nez} dP + \frac{f}{nee} dR \left| \frac{R + T}{S} = \frac{a^3}{yy} \right.$

$\sqrt{yy - aa} + aa \left| \frac{1 + \sqrt{yy - aa}}{y} \right.$, by substituting

$\sqrt{yy - aa}$ for P ($\sqrt{e + fz^n}$), a for R ($\sqrt{e}$), $\sqrt{yy - aa}$ for T ($\sqrt{yy - aa}$), and y for S ($\sqrt{fz^n}$).

And that against $\theta = 0$, is $\frac{-2}{ne} dR \left| \frac{R + T}{S} = \right.$

$-aa \left| \frac{1 + \sqrt{yy - aa}}{y} \right.$, by substituting the same

Values. Therefore the Sum of these Fluents

$$\frac{a^3}{yy} \sqrt{yy - aa} + aa \left| \frac{1 + \sqrt{yy - aa}}{y} \right. - aa \left| \frac{1 + \sqrt{yy - aa}}{y} \right. \text{ is } = \frac{a^3}{yy} \sqrt{yy - aa} \text{ (since the}$$

Measures

Measures of the same Angle to the Module aa are, the one affirmative, and the other negative, and consequently destroy each other.) = Fluent of the Fluxion of the Lunar Space $EPQB$, = $EPQB$.

Now this Fluent may be easily constructed; for if a right Line be drawn from the Centre C to P , the Isosceles right-lined Triangle CPE will be = Fluent $\frac{a^3}{yy} \sqrt{yy - aa} = \frac{a^3}{yy} x$ (since $x = \sqrt{yy - aa}$) = Lunar Space $EPQB$, as may be shewn thus:

Draw PH parallel to GC ; then the Triangles AGC , APH are similar: therefore $AG(y) : AP \left(\frac{2aa}{y} \right) : GC(x) : PH = \frac{2aax}{yy}$.

And so $\frac{1}{2} PH \left(\frac{aax}{yy} \right) \times CE(a)$ is = $\frac{a^3x}{yy}$ = Fluent to be constructed = Area of the Triangle CPE . Hence the Area of the Semi-Lune is = $CE \times \frac{1}{2} DC = \frac{1}{2} aa$. And thus you have the Quadrature of the aforesaid Space by the Method of Fluxions; tho' indeed it may be shewn much shorter by common Geometry.

EXAMPLE XVI.

34. **T**O square the Cycloidal Space, or to find the Area of any Segment AMG of it.

Let APB be the generating Circle. Let AP be any Ordinate, PM the correspondent Absciss; let mp be infinitely near MP ; let PM touch the generating Circle in P , and MT the Cycloid in M . Then from the Nature of the Cycloid, the Subtangent $PT = PM = \text{Arch } AP$. Draw AG perpendicular to Dd to

FIG. 26.

to AB , and from the Points M, m , draw MG, mg perpendicular to AG .

Now let $AQ = x$, $AB = 1$. Because $TP = PM$, the Angle $MTP = PMT$; and therefore the Angle $TPQ = 2TMP$. But the Measure of the Angle APQ is $\frac{1}{2}$ the Arch AP , which is also the Measure of the Angle TPA ; therefore $APQ = TMP = Mms$. Consequently the Triangles APQ, Mms are similar, therefore $AQ(x) : QP(\sqrt{x-xx}) :: MS(\dot{x}) : mS = \dot{x} \frac{\sqrt{x-xx}}{x}$: But $\dot{x} \frac{\sqrt{x-xx}}{x}$

thrown into an infinite Series, will be $x^{-\frac{1}{2}}$
 $\dot{x} - \frac{1}{2}x^{\frac{3}{2}}\dot{x} - \frac{1}{8}x^{\frac{5}{2}}\dot{x} - \frac{1}{16}x^{\frac{7}{2}}\dot{x}$, &c. = the Fluxion of the Ordinate QM to the Axis AB of the Cycloid, and the Fluent of this, viz.
 $2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{25}x^{\frac{5}{2}} - \frac{1}{128}x^{\frac{7}{2}}$, &c. will be the said Ordinate MQ . Whence $QM \times \dot{x}$ = the Fluxion of the Cycloidal Space AMQ is =
 $2x^{\frac{3}{2}}\dot{x} - \frac{1}{3}x^{\frac{5}{2}}\dot{x} - \frac{1}{25}x^{\frac{7}{2}}\dot{x} - \frac{1}{128}x^{\frac{9}{2}}\dot{x}$, &c. the Fluent of which will be $\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} - \frac{1}{2112}x^{\frac{9}{2}}$, &c. = Area of the indefinite Cycloidal Space AMQ .

Now if $mS = gG = \frac{\dot{x}\sqrt{x-xx}}{x}$ be drawn into $GM = AQ = x$; then will the Fluxion $GMSG$ of the Area AMG be equal to $\dot{x}\sqrt{x-xx}$. Therefore since this is the same as the Fluxion of the Segment of the Circle APQ , the Space AMG will be equal to the Segment APQ of the Circle, and so the Area $ADBC$ of the whole Cycloidal Space ADB is equal to the Area of the Semicircle APB .

COROL.

COROL.

BECAUSE DB is equal to $\frac{1}{2}$ the Circumference of the Circle; if you call the same p , and AB, a ; then the Rectangle $ABDE = ap = \text{Area Semicircle } APB$. Whence the external Cycloidal Space $AEDMA = \frac{1}{4} ap$. Therefore the Area of the Semicycloid $ADB = \frac{3}{4} ap$ $AMDPA = \frac{1}{4} ap$. Consequently the Area of the Cycloid is the triple of the generating Circle.

EXAMPLE XVII.

35. *TO square the Cissoïd of Diocles, or to find the Area of any Segment APM of it.* FIG. 27.

Let ADB be the generating Circle, BH the Asymptote to the Curve (AI) of the Cissoïd, at right Angles to the Diameter AB ; let the Diameter AB be $= 1$, the Absciss $AP = x$, the correspondent Ordinate PM to the Cissoïd $= y$.

Now the Equation expressing the Nature of this Curve will be $PB \times PM^2 = AP^3$; that is, $y^2 - xy^2 = x^3$, and so $y^2 = \frac{x^3}{1-x}$. Whence

$$y = \sqrt{\frac{x^3}{1-x}} = \frac{x\sqrt{x}}{\sqrt{1-x}} = x^{\frac{3}{2}} \times 1-x^{-\frac{1}{2}}; \text{ and } xy = x^{\frac{3}{2}} \times 1-x^{-\frac{1}{2}} = \text{Fluxion of the Area } APM.$$

The Fluent of which will be $\frac{2}{5} x^{\frac{5}{2}} + \frac{2x^{\frac{7}{2}}}{2.7} +$

$$\frac{1.3x^{\frac{9}{2}}}{4.9} + \frac{1.3.5x^{\frac{11}{2}}}{4.6.11} + \frac{1.3.5.7x^{\frac{13}{2}}}{4.6.8.13}, \text{ \&c.} = \sqrt{x} \text{ in-}$$

to $\frac{2}{5}x^4 + \frac{1}{7}x^6 + \frac{1.3x^8}{4.9} + \frac{1.3.5x^{10}}{4.6.11} + \frac{1.3.5.7x^{12}}{4.6.8.13}$
 $\mathcal{E}c. = \text{Space } APM.$ And when $x=1$, then
 will this Series become $\frac{2}{5} + \frac{1}{7} + \frac{1.3}{4.9} + \frac{1.3.5}{4.6.11}$
 $+ \frac{1.3.5.7}{4.6.8.13}, \mathcal{E}c. = \text{Area of the infinite Space}$
 $ABHIA.$

Otherwise:

Let $AB=a$, and $PN=\dot{z}$. Now because
 from the Nature of the Curve $ay^2 - xy^2 = x^3$;
 this thrown into Fluxions, and there will arise
 $2ay\dot{y} - 2xy\dot{y} - y^2\dot{x} = 3x^2\dot{x}$, and $2a - 2x \times \dot{y} =$
 $y\dot{x} = \frac{3x^2\dot{x}}{y}$. But since from another Property
 of the Curve $x^2 = zy$; therefore $\frac{x^2}{y} = z$. Now
 make $a - x = PB = u$; then we shall have $2u\dot{y}$
 $- y\dot{x} = 3z\dot{x}$; and so the Fluent of the one
 will be equal to the Fluent of the other. But
 zx is the Fluxion $PNnp$ of the Segment ANP
 of the Circle; and because $u = PB = OM$ and
 $\dot{y} = mR = Oo$, $mMOo$, will be the Fluxion
 of the Area $AMOB$, and $y\dot{x}$ the Fluxion of
 the Area AMP . Now when the Fluent of
 $u\dot{y}$ expresses the Area of the whole Cycloidal
 Space $ABHIA$, the Fluent of $x\dot{y}$ will be the
 same Area; and so the Fluent of $2u\dot{y} - y\dot{x} =$
 Fluent $u\dot{y}$. Therefore since in the same Case
 the Fluent of zx is the Area of the Semicircle
 ANB ; and because the Fluent of $u\dot{y} =$
 $3zx$, the whole Cissoidal Space $ABHIA$
 will be the triple of the generating Semicircle
 $ANB.$

EXAM.

EXAMPLE XVIII.

36. **T**O square any Interminate Space $HPMI$ FIG. 28.
 contained under the Asymptote PH , Or-
 dinate MP , and Part MSI of the Logarith-
 mick Curve MI .

Call the Subtangent PT, a (because from the Nature of this Curve it is a standing Quantity) and the Ordinate PM, y . Draw the infinitely near Ordinate pm , and from M draw Mq perpendicular to mq . Now because of the similar Triangles $MTP, mMq, MP(y) : TP(a) :: mq(\dot{y}) : Mq = \frac{a\dot{y}}{y}$. Whence MP

$\times Mq = y \times \frac{a\dot{y}}{y} = a\dot{y}$, is the Fluxion of the Area $IMPH$, and the Fluent thereof will be ya ; that is, the Area of the infinitely extended Space $IMPH$ is $= MP \times PT$.

COROL.

IF the Ordinate QS be $= z$; then the Interminate Space $ISQH = az$, and consequently $SM PQ = ay - az = a \times y - z$; that is, the Space contained under any two Semiordinates MP, SQ , the Part of the Absciss PQ , and the Curve MS , is equal to $TP \times MP - SQ$; and so the Space $BAPM$ is to the Space $BMSQ$, as the Difference of the Ordinates AB, PM to the Difference of the Ordinates PM, QS .

EXAMPLE XIX.

37. *TO Square Spiral Spaces.*

FIG. 29. Let the Semi-diameter of the Circle, viz. $AC = a$, the Periphery $= b$, any Arch $AB = x$, as an Absciss, and the correspondent Ordinate $CM = y$. Conceive the Radius Cb infinitely near CB , and draw the small Arch ME .

Now the Nature of *Archimedes's* Spiral is $AC \times AB = \text{Periphery } b$ drawn into CM ; that is, $ax = by$. This being granted, the small Arch $ME = \frac{y \dot{x}}{a}$, since $CB : Bb :: CM : ME$.

Therefore $\frac{1}{2} CM \times ME = \text{Area of the little Sector } MCE = \text{to the little trilineal Space } CMm$, which is the Fluxion of the Spiral Space $= \frac{y^2 \dot{x}}{2a}$; but from the Nature of the Curve $ax = by$; therefore $\frac{a^2 x^2}{b^2} = yy$. Consequently substituting $\frac{a^2 x^2}{b^2}$ for yy in the Fluxionary Expression, and then it will be $\frac{ax^2 \dot{x}}{2b^2}$;

the Fluent of which will be $\frac{ax^3}{6b^2}$, the Area of the Segment of the Spiral Space; and if for x be put b , the whole Circumference, the whole Spiral Space will be $= \frac{1}{6} ab^3$.

Again; The Nature of all Kinds of circular Spirals will be expressed by $a^m x^n = b^n y^m$; therefore $\frac{a^m x^n}{b^n} = y^m$, and $\frac{a x^{\frac{n}{m}}}{b^{\frac{n}{m}}} = y^2$. Conse-

quently $\frac{y^2}{2a} \dot{x} = \frac{a x^{\frac{2n}{m}} \dot{x}}{2b^{\frac{2n}{m}}}$. The Fluent of which

I

will

will be $\frac{max^{\frac{2n+m}{m}}}{4n+2m \times b^{\frac{2n}{m}}}$. Therefore putting b for x , and the whole Spiral Spaces will be $\frac{mab}{4n+2m}$.

Moreover, if the Arch AB be to BM as the Absciss to the Ordinate in any Algebraical Curve, the Spiral Space may be squared after the same manner as above.

For Example: Let AB be to BM as the Absciss of a Parabola to the Ordinate; then assuming p for the Parameter $px = a^2 - 2ay + yy$, and $\dot{x} = \frac{2y\dot{y} - 2a\dot{y}}{p}$. Whence $\frac{y^2\dot{x}}{2a} = \frac{y^3\dot{y} - ay^2\dot{y}}{ap}$, the Fluent of which will be $\frac{y^4}{4ap - y^3}$.

Much after the same manner you may find the Area of the Space contained under the Arch AB , and the Spiral AM ; whose Fluxion is the Trapezium $BMmb = Bb + Mm \times \frac{1}{2}mb$.

But $Bb = \dot{x}$, $Mm = \frac{y\dot{x}}{a}$, $mb = a - y$; there-

fore $BMmb \left(\dot{x} + \frac{y\dot{x}}{a} \times \frac{1}{2}a - y \right) = \frac{a^2\dot{x} - y^2\dot{x}}{2a}$.

Now let the Curve be a Parabolical Spiral; substitute $\frac{2y\dot{y} - 2a\dot{y}}{p}$ for its Equal $\dot{x}$: then will

$\frac{ay^2\dot{y} + a^2y\dot{y} - y^3\dot{y} - a^3\dot{y}}{ap}$ be the Fluxion of the Space ABM , and the Fluent thereof y^3 .

$$\frac{\frac{y^3}{3p + ay^2}}{2p - y^4} \text{ will be equal to the said Area.}$$

$$\frac{4ap - a^2y}{p}$$

EXAMPLE XX.

FIG. 30. 38. **T**O square any Quadratical Space *ACMP* contained under the Base *AC*, the Absciss *AP*, a correspondent Ordinate *PM*, and Part *CM* of the Quadratrix.

Let *Cc* be a Quadrant of the generating Circle, *A* being the Centre. Draw *AmM*, and from *m* and *M* let fall the Perpendiculars *me*, *ME*. Now let *AC* = 1, *AP* = *x*, *PM* = *y*. Then the Nature of this Curve is, that *AB* (*x*) = *EM* is = Arch *Cm* of the Quadrant; therefore (by *Schol.* 1. *Example* 4. of the next Section) the Sine *em* will be = $1 + x - \frac{1}{2}x^3 + \frac{1}{120}x^5$, &c. and the Sine Complement *Ae* = $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$, &c. But because the Triangles *Ame*, *APM* are similar, therefore *me* ($1 + x - \frac{1}{2}x^3 + \frac{1}{120}x^5$, &c.) : *Ae* ($1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$, &c.) :: *AP* (*x*) : *PM* (*y*) = $1 - \frac{1}{2}x^2 - \frac{1}{45}x^4 - \frac{2}{345}x^6$, &c.) Whence $y\dot{x} = \dot{x} - \frac{1}{2}x^2\dot{x} - \frac{1}{45}x^4\dot{x} - \frac{2}{345}x^6\dot{x}$, &c. = Fluxion of the Area *ACMP*, and the Fluent will be = $x - \frac{1}{2}x^3 - \frac{1}{135}x^5 - \frac{2}{2075}x^7$, &c. = Area *ACPM*.

SECT.



SECTION IV.

Use of Fluxions in the Rectification of Curves.

P R O B.

39. **T**O rectify any given Curve, or find a strait Line equal to the same.

Since a Curve is conceived to be made up of an infinite Number of small right Lines; one of them being found by the direct Method of Fluxions, that is, the Fluxion of the Curve; and the Sum of them, or, which is all one, the Fluent of that Fluxion, will be the Length of a right Line equal to the Curve. Now let $PM(y)$ be the Ordinate at right Angles to the Absciss $AP(x)$ of any Curve AM , and let TM touch the Curve in M . Draw Pm parallel to PM and infinitely near PM , and from M draw Mm perpendicular to pm . Then $Pp(=Mn)=\dot{x}$, and $nm=\dot{y}$, and Mm will be the Fluxion of the Curve AM ; which must first be found, either by getting the Value of $\overline{Mn}^2(\dot{x}^2)$, or $\overline{mn}^2(\dot{y}^2)$ in the Equation of the Curve thrown into Fluxions, because $\dot{x}^2 + \dot{y}^2 = Mm$; or by making the Subtangent TP , or Ordinate $PM(y)$ to the Tangent TM , as the Fluxion of the Absciss $Mn(\dot{x})$

FIG. 31.

E c

or

or $mn (\dot{y})$ to a fourth Proportional, which will be $= Mn$ the Fluxion of the Curve AM . For the little right-angled Triangle Mmn is similar to the right-angled Triangle TMP ; and the Fluent of that Fluxion will be the Arch sought. Some Examples will make this evident.

C O R O L.

HENCE if PQ be drawn from P perpendicular to the Tangent TM ; PQ or QM : $PM (\dot{y}) :: Mn (\dot{x})$, or $nm (\dot{y}) : Mm$ the Fluxion of the Curve AM .

E X A M P L E I.

40. TO find the Length of any Arch AM of the Curve of the common Parabola.

FIG. 31. Here $AP \times a = \overline{PM}^2$; that is, $ax = yy$; and both Parts thrown into Fluxions is $a\dot{x} = 2y\dot{y}$. Whence $aa\dot{x}^2 = 4y^2\dot{y}^2$, and $\dot{x}^2 = \frac{4y^2\dot{y}^2}{aa}$. And adding $\dot{y}^2$ to this last Expression, we shall have $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\dot{y}^2 + \frac{4y^2\dot{y}^2}{aa}} = \frac{\dot{y}}{a} \sqrt{aa + 4yy} = Mm$ the Fluxion of the Curve AM . The Fluent of which will be $y + \frac{2y^3}{3a^2} - \frac{2y^5}{5a^4} + \frac{4y^7}{7a^6} - \frac{10y^9}{9a^8}$ &c. = Arch AM .

Otherwise by the Measure of a Ratio.

The said Fluxion of the Curve of the Parabola $\frac{\dot{y}}{a} \sqrt{aa + 4yy}$ may be referred to the fourth Form in the Tables of Mr. Cotes. For if

if you make $z=y$, $n=2$, $\theta=0$, $d=\frac{1}{a}$, $e=aa$,

$f=4$. Then will $dz z^{\theta n + \frac{1}{2}n - 1} \sqrt{e + fz^n}$ be $= \frac{1}{2} \sqrt{aa + 4yy}$. And the Fluent of this Form

(against $\theta=0$) is $\frac{z^n}{n} dP + \frac{e}{nf} dR \left| \frac{R+T}{S} \right.$. And

making $P (= \sqrt{\frac{e + fz^n}{z^n}}) = \frac{1}{y} \sqrt{aa + 4yy}$, R

$(= \sqrt{f}) = 2$, $T (= \sqrt{\frac{e + fz^n}{z^n}}) = \frac{1}{y} \sqrt{aa + 4yy}$,

$S (= \sqrt{\frac{e}{z^n}}) = \frac{a}{y}$. The said Fluent will be $\frac{y}{2a}$

$\sqrt{aa + 4yy} + \frac{a}{4} \left| \frac{\sqrt{aa + 4yy} + 2y}{a} \right.$. Which may

be thus constructed: From the Vertex A draw AB bisecting the Ordinate PM (y) in B ; then FIG. 32.

will $AB (= \sqrt{AP^2 + PB^2}) = \frac{y}{2a} \sqrt{aa + 4yy}$.

And multiplying the Numerator and Denomi-

nator of the Ratio $\frac{\sqrt{aa + 4yy} + 2y}{a}$ by $\frac{y}{2a}$; the

same will become $\frac{y}{2a} \sqrt{aa + 4yy} + \frac{yy}{a}$. That is,

$$\frac{a}{4} \left| \frac{\sqrt{aa + 4yy} + 2y}{a} \right. \text{ is } = \frac{a}{4} \left| \frac{\frac{y}{2a} \sqrt{aa + 4yy} + \frac{yy}{a}}{\frac{1}{2}y} \right.$$

But $\frac{y}{2a} \sqrt{aa + 4yy} + \frac{yy}{a}$ is $= AB + BP$, and

$\frac{1}{2}y$ is $= PB$; also $\frac{a}{4}$ is $=$ focal Distance from

the Nature of the Parabola. Therefore

Ee 2

$\frac{a}{4}$

$$\frac{a}{4} \left| \frac{\sqrt{aa + 4yy} + 2y}{a} \right| \text{ is } = AF \left| \frac{AB + AP}{PB} \right| = \text{to}$$

the Measure of the Ratio between $AB + AP$, and PB to the focal Distance AF , as a Module. This added to AB , and the whole will

$$\text{be } \frac{y}{2a} \sqrt{aa + 4y^2} + \frac{a}{4} \left| \frac{\sqrt{aa + 4yy} + 2y}{a} \right| = \text{Flu-}$$

$$\text{ent of the Fluxion } \frac{\dot{y}}{a} \sqrt{aa + 4yy} = \text{Length}$$

of the Arch AM of the Parabola. Hence the Rule for finding the Length of the Curve of the common Parabola is this.

Let A be the Vertex, F the Focus, AP the Axis, and PM an Ordinate to the same. Draw AB bisecting the Ordinate PM in the Point B ; to which continued out, add BC the Measure of the Ratio between $AB + AP$, and PB to the Module AF ; and then AC will be the Length of the Arch AM of the Parabola. This is the Construction given by Mr. Cotes in *Harmonia Mensurarum*, p. 12.

In this Curve the Subtangent $TP = 2AP = 2x$. Whence $TM = \sqrt{4xx + ax}$. Therefore $TP(2x) : TM(\sqrt{4xx + ax}) :: Mn(x) : Mm$
 $= \frac{\sqrt{4xx + ax}}{2x} \times \dot{x} = \text{Fluxion of the Curve.}$

Or since $ax = yy$; therefore $x = \frac{yy}{a}$, and $2x$

$$= \frac{2yy}{a}, \text{ and } 4xx = \frac{4y^4}{aa}. \text{ Consequently } \frac{\sqrt{4y^4 + yy}}{aa}$$

$$= TM. \text{ Now } PM(y) : TM\left(\frac{\sqrt{4y^4 + yy}}{aa}\right)$$

$$:: nm(\dot{y}) : Mm = \dot{y} \frac{\frac{\sqrt{4y^4 + yy}}{aa}}{y} = \dot{y} \frac{\sqrt{aa + 4yy}}{a}$$

as before; and the Fluent of $x \sqrt{\frac{4xx + ax}{2x}}$,

supposing $a=1$, will be $x^{\frac{1}{2}} + \frac{2}{3} \sqrt{x^3} - \frac{2}{5} \sqrt{x^5}$
 $+ \frac{4}{7} \sqrt{x^7} - \frac{2}{9} \sqrt{x^9}$, &c.

C O R O L.

IF AC, DC be the Semi-conjugate Axes of FIG. 32.
 an Equilateral Hyperbola, and $AC (=DC)$
 be supposed $=a$, the *Latus Rectum* of the
 Parabola; and the Ordinate PM be $=2y$, and
 the Absciss $QM=x$: then will AP be $=x-a$.

Consequently because $PC \times AP = \overline{PM}^2$; that
 is $xx - aa = 4yy$, and so $xx = 4yy + aa$. There-
 fore $x = \sqrt{4yy + aa}$. And if qm be drawn
 infinitely near qm , then $Qq=y$; and so the
 Fluxion $QqmM$ of the Area of the Hyperbo-
 lick Space $CQMA$ will be $=\dot{y} \sqrt{aa + 4yy}$
 $=$ Fluxion of the Curve of the Parabola.
 Therefore the Quadrature of the Curve of the
 Parabola depends upon the Quadrature of the
 said Hyperbolic Space. And so the Rectifi-
 cation of any Curve may be brought to the
 Quadrature of a Curve, by supposing the Flu-
 xion of the Curve to be rectify'd (found as a-
 bove) as an Ordinate, and the variable Quan-
 tity in that Fluxion as an Absciss to that Or-
 dinate. Consequently the Business of rectify-
 ing Curves sometimes may be shorten'd, from
 a Pre-knowledge of the Quadrature of that
 Curve it may be reduced too.

EXAM-

EXAMPLE II.

41. **T**O rectify a Parabola of the second Kind; where $ax^2 = y^3$, or making $a = 1$, $x^2 = y^3$.

Because $x^2 = y^3$, therefore $2x\dot{x} = 3y^2\dot{y}$, and $4x^2\dot{x}^2 = 9y^4\dot{y}^2$. Whence $\dot{x}^2 = \frac{9y^4\dot{y}^2}{4x^2} = \frac{9y^4\dot{y}^2}{4y^3}$ (by substituting y^3 for ax^2) $= \frac{9}{4}y\dot{y}^2$; therefore $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\frac{9}{4}y\dot{y}^2 + \dot{y}^2}$ (by adding $\dot{y}^2$ to each Side of the Equation $\dot{x}^2 = \frac{9}{4}y\dot{y}^2$) $= \frac{1}{2}\sqrt{9y\dot{y}^2 + 4\dot{y}^2} = \frac{1}{2}\dot{y}\sqrt{9y + 4}$, = Fluxion of the Curve, and the Fluent thereof will be $\frac{1}{27} \times 9y + 4 \times \sqrt{9y + 4}$. But in order to find whether any thing is to be added or taken from this, make $y = 0$, and then the Remainder will be $\frac{4}{27}\sqrt{4} = \frac{8}{27}$; therefore the Length of the Curve will be * $\frac{1}{27} \times 9y + 4 \times \sqrt{9y + 4} - \frac{8}{27}$.

* Art. 19.

COROL.

FIG. 31. **L**ET the Parameter of the common Parabola be 1, $AP = 1$, $PQ = \frac{1}{4}y$; then will $AQ = \frac{1}{4}y + 1$, and because the Parameter is 1, $QN^2 (\frac{1}{4}y + 1) = \frac{9y + 4}{4}$. Consequently $QN = \frac{1}{2}\sqrt{9y + 4}$; therefore the Fluxion $QNNq$ of the Parabolick Space $PMNQ$ is $= \frac{1}{2}\dot{y}\sqrt{9y + 4}$. And so the Length of the Curve of a Parabola of the second Kind expressed by $ax^2 = y^3$ depends on, or is the Quadrature of the common Parabola, and may be had in finite Terms.

EXAMPLE III.

42. **T**O rectify Parabola's of all Kinds.

Let the Parameter be $=1$; then the Nature of infinite Numbers of Parabola's of different Kinds will be expressed by this Equation $y^m = x$. The Fluxion of which will be $my^{m-1}y = \dot{x}$; and squaring both Sides $m^2 y^{2m-2} y^2 = \dot{x}^2$. And if for Brevity's sake you make $2m-2=r$, then will $m^2 y^r y^2 = \dot{x}^2$; and adding y^2 to each Side, and afterwards extracting the square Root, $\sqrt{\dot{x}^2 + y^2} = \sqrt{m^2 y^r y^2 + y^2} = y \sqrt{m^2 y^r + 1}$ = Fluxion of the Arch of a Parabola of any Kind soever. The Fluent of which, viz.

$$y + \frac{m^2 y^{r+1}}{2.r+1} - \frac{m^4 y^{r+1}}{2.4.2r+1} + \frac{1.3m^6 y^{r+1}}{2.4.6.3r+1} - \frac{1.3.5m^8 y^{r+1}}{2.4.6.8.4r+1}, \text{ \&c. will be the Length of}$$

the Curve thereof. Now substituting $2m-2$ for r , and then the same Arch will be =

$$y + \frac{m^2 y^{2m-1}}{2.2m-1} - \frac{m^4 y^{2m-1}}{2.4.4m-3} + \frac{1.3m^6 y^{2m-1}}{2.4.6.6m-5} - \frac{1.3.5m^8 y^{2m-1}}{2.4.6.8.8m-7}, \text{ \&c.}$$

EXAM-

EXAMPLE IV.

43. *TO find the Length of any Part BC of the Equiangular or Logarithmical Spiral.*

FIG. 33. Let the Radius $AB=a$, $AC=b$: Let BD (c) touch the Spiral in B , and AD be perpendicular to BD . Call BD , p . Also let PE touch the Spiral in P , and let AE be perpendicular to it. Moreover let Ap be infinitely near AP ; and from A describe the small Arch Pm . Call $AP-AB=FP$, y .

Now since from the Nature of this Curve any Radius AB cuts it in the same Angle, or makes the same Angle ABD with the Tangent BD ; therefore any Triangle, as APE , will be similar to the given Triangle ABD . But the small Triangle mpP is also similar to the Triangle APE , the Angle at m being a right Angle $=E$, and the Angle at p common to both Triangles: whence the fluxionary Triangle mpP is similar to the Triangle ABD ; and so $BD(c):AB(a)::mp(\dot{y}):Pp\left(\frac{a\dot{y}}{c}\right)=$ Fluxion of the Part BP of the Curve; and the Fluent $\frac{ay}{c}$ will be the Length of the Part BP ; and putting $b-a$ for y , we shall have $\frac{ba-aa}{c} =$ Length of BC ; that is, as BD (c): $AB(a)::AC-AB$: Length BC .

EXAMPLE V.

44. **T**O find the Length of any Arch AP of the Spiral APC of Archimedes.

Let CAM be the generating Circle. Call FIG. 34. AC, a , the Circumference b , the Arch CM, x , the Ordinate AP of the Spiral y , and let PT touch the Spiral in P ; intersecting the Line AT drawn from A the Centre of the Spiral perpendicular to AP , in the Point T . Moreover, let $Mm = \dot{x}$, draw Am , and with the Distance An , from A describe the little Arch np . Now $ax = by$ expresses the Nature of this Curve. Because of the similar Sectors AMm , Apn , $AM(a) : Mm(\dot{x}) :: Ap = AP(y) : Pn = \frac{y\dot{x}}{a}$. And since the Triangles Ppn , PAT are similar also; therefore $pP(\dot{y}) : pn\left(\frac{y\dot{x}}{a}\right) :: AP(y) : AT$ the Subtangent $= \frac{yy\dot{x}}{a\dot{y}}$. But $a\dot{x} = b\dot{y}$; whence $\dot{x} = \frac{b\dot{y}}{a}$. Which substituted for $\dot{x}$ in $\frac{yy\dot{x}}{a\dot{y}}$, and we have $AT = \frac{byy}{aa}$. Again, since TAP is a right Angle, $\overline{AT}^2 \left(\frac{bby^4}{a^4}\right) + \overline{AP}^2(yy) = \overline{TP}^2 = \frac{a^4yy + bby^4}{a^4}$; and so $TP = \frac{y}{aa} \sqrt{a^4 + bbyy}$; and $AP(y) : TP\left(\frac{y}{aa} \sqrt{a^4 + bbyy}\right) :: pP(\dot{y}) : Pn = \frac{1}{aa} \dot{y} \sqrt{a^4 + bbyy}$, = Fluxion of the Part AP of the Spiral. Which comes under the fourth Form of Mr. Cotes's Tables.

For making $z=y$, $n=2$, $\theta=0$, $d=\frac{1}{aa}$, $e=a^4$, $f=bb$, we have $dz z^{\theta n + \frac{1}{n} - 1} \sqrt{e + fz^n} = \frac{1}{aa} y \sqrt{a^4 + bbyy}$. The Fluent of which is $\frac{z^n}{n} dP + \frac{e}{nf} dR \left| \frac{R+T}{S} \right.$. And writing for P , R , T , S , or $\sqrt{\frac{e+fz^n}{z^n}}$, $\sqrt{f}$, $\sqrt{\frac{e+fz^n}{z^n}}$, $\sqrt{\frac{e}{z^n}}$, their Values $\frac{1}{y} \sqrt{a^4 + bbyy}$, b , $\frac{1}{y} \sqrt{a^4 + bbyy}$, $\frac{a^2}{y}$, there arises $\frac{y}{2aa} \sqrt{a^4 + bbyy} + \frac{aa}{2b} \left| \frac{\sqrt{a^4 + bbyy} + yb}{aa} \right. =$ Fluent of the Fluxion aforesaid.

This Fluent may be constructed thus: Bisect AP (y) in the Point L , and draw DL parallel to the Tangent PT cutting the Subtangent in D ; then will DL be $= \frac{y}{2aa} \sqrt{a^4 + bbyy}$,

since it is $\frac{1}{2} PT = \frac{y}{aa} \sqrt{a^4 + bbyy}$. Again, the Module $\frac{aa}{2b}$ is had by making $AD \left(\frac{byy}{2aa} \right)$:

$AL \left(\frac{y}{2} \right) :: AL \left(\frac{y}{2} \right) : AF = \frac{aa}{2b}$. And

the Ratio $\frac{\sqrt{a^4 + bbyy} + yb}{aa}$ is =

$\frac{\frac{y}{2aa} \sqrt{a^4 + bbyy} + \frac{by^3}{2aa}}{\frac{y}{2}}$, as you will find by di-

viding the Numerator and Denominator of this last by $\frac{y}{2}$; therefore that Ratio is $= \frac{LD + AD}{AL}$. Consequently if to DL you add

LQ ,

LQ , which is the Measure of the Ratio of $LD + AD$ to AL , the Line DF being the Module; the whole Line DQ will be equal to the Length AP of the Spiral; and thus you may get the Length of the whole Spiral by making $y=a$, the Radius of the generating Circle.

E X A M P L E VI.

45. **T**O find the Length of any Arch CP of Fig. 35. the reciprocal Spiral APC .

About the Centre A , with the Distance AC , describe the Quadrantal Arch $BCMH$, and continue out AP to M ; then the Nature of this Curve is, that any Radius AC of it is reciprocally as the Angle BAC it makes with the first Radius AB , or as the Arch BC ; that is, $AP : AC :: BC : BM$; and so $AP \times BM = AC \times BC$.

Now make AB or $AC=a$, the Arch $BC=b$, the Arch $BM=x$, and $AP=y$; then will ab be $=xy$. Draw PG to touch the Curve in P , and at right Angles to AP draw AG . Let Ap be infinitely near AP , and continue it out to intersect the Arch in m , and with the Distance Ap from A describe the small Arch pn . Because the Sectors Anp , AMm are similar, $AM(a) : Mm(\dot{x}) :: AP(y) : pn = \frac{\dot{x}y}{a}$. And because of the similar Triangles APG , nPp , therefore $Pn(\dot{y}) : pn\left(\frac{\dot{x}y}{a}\right) :: AP(y) : AG = \frac{\dot{x}y^2}{ay}$. And throwing the Equation of the Curve $ab=xy$ into Fluxions, we have $\dot{x} = \frac{a\dot{b}y}{yy}$, which put for $\dot{x}$ in $\frac{\dot{x}y^2}{ay}$, and

we have $AG=b$; that is, the Line drawn from the Centre A perpendicular to any Radius AP of the Spiral, to intersect the Tangent to the Spiral at the Extremity of that Radius, will be a standing Quantity, viz. = Arch BC , which Arch will become a straight Line, perpendicular to the first Radius AB , when the Point C is at an infinite Distance, and the Angle BAC infinitely small; that is, it will be equal to AD the Distance of the Asymptote DE from the Centre. Hence $AP(y):GP$

$(\sqrt{bb+yy})::nP(\dot{y}):Pp=\frac{\dot{y}}{y}\sqrt{bb+yy}=\text{Fluxion of the Curve } AP.$

This may be referred to the third Form in the Tables of Mr. Cotes. For making $z=y$, $n=2$, $\theta=0$, $d=1$, $e=bb$, $f=1$, we have $d\dot{z}z^{\theta n-1}\sqrt{e+fz^n}=\frac{\dot{y}}{y}\sqrt{bb+yy}$. And the

Fluent of this $\frac{2}{n}dP-\frac{2}{n}dR\left|\frac{R+T}{S}\right|$, by writing for $P(=\sqrt{e+fz^n})\sqrt{bb+yy}$, for $R(=\sqrt{e})b$, for $T(=\sqrt{e+fz^n})\sqrt{bb+yy}$, and for $S(=\sqrt{fz^n})y$, will be $=\sqrt{bb+yy}-b\left|\frac{b+\sqrt{bb+yy}}{y}\right|$.

And if AC be supposed invariable $=z$, then the Fluent $\sqrt{bb+zz}-b\left|\frac{b+\sqrt{bb+zz}}{z}\right|$ will be $=\text{Arch } APC$. And the Difference of these

Fluents, viz. $\sqrt{bb+zz}-\sqrt{bb+yy}+b\left|\frac{b+\sqrt{bb+zz}}{z}\right|-b\left|\frac{b+\sqrt{bb+yy}}{y}\right|=CF-PG+AF\left|\frac{AF+PG}{AP}\right|-AF\left|\frac{AF+CF}{AC}\right|=\text{Length of the Part } PC \text{ of the Curve; that is, if to the}$

the Difference $CF - PG$ of the Tangents, you add the Difference of the Measures of the Ratio between $AF + PG$ and AP to the Module AF , and of the Ratio between $AF + CF$ and AC to the same Module AF , the whole will be the Length of the Part PC of the Curve; or because $AF \left| \frac{AF + PG}{AP} \right|$ is = to $AF \left| \frac{AP}{PG - AF} \right|$, and $AF \left| \frac{AF + CF}{AC} \right|$ equal $AF \left| \frac{AC}{CF - AF} \right|$ (by *Schol. Art. 14.*) therefore if LM be the Measure of the Ratio between AC and $CF - AF$, to the Module AF , and in like manner lm the Measure of the Ratio between AP and $PG - AF$ to the Module AF ; then the Aggregate of the Difference of the Tangents $CF - PG$, and the Difference of the Measures $lm - LM$ will be = Length of the Part PC of the Spiral.

EXAMPLE VII.

46. **T**O find the Length of the Arch CP of FIG. 36. the Logarithmick Curve CPG .

Let AC, aP be Ordinates perpendicular to the Asymptote Af , and let CF, Pf be Tangents. Also let n be infinitely near P , and draw np parallel to af .

Now call AC, z, aP, y , and the invariable Subtangent AF or af, b . Then because of the similar Triangles aPf, nPp , we have $aP (y)$

$$: fP (\sqrt{bb + yy}) :: nP (y) : Pp = \frac{y}{b} \sqrt{bb + yy} =$$

Fluxion of the infinite Part PG of the Curve.

Which compared with the third Form of Mr. Cotes's Tables, and the Fluent (as in the last Problem) will be $\sqrt{bb+yy}-b \left| \frac{b+\sqrt{bb+yy}}{y} \right|$.

In like manner the Fluxion of the infinite Part CPG will be $\frac{\dot{z}}{z} \sqrt{bb+zz}$; and the Fluent

thereof will be $\sqrt{bb+zz}-b \left| \frac{b+\sqrt{bb+zz}}{z} \right|$;

and the Difference of these Fluents, viz.

$\sqrt{bb+zz}-\sqrt{bb+yy}+b \left| \frac{b+\sqrt{bb+zz}}{z} \right| - b$

$\frac{b+\sqrt{bb+yy}}{y} = CF - Pf + AF \left| \frac{AF + Pf}{AP} \right|$

* Art. 14. $-AF \left| \frac{AF + CF}{AC} \right|$, or * $= CF - Pf + AF$

$\left| \frac{aP}{Pf - AF} - AF \left| \frac{AC}{CF - AF} \right| \right|$ Now if AL

be taken equal to $CF - AF$, and al equal to

$Pf - AF$, and LM, lm be drawn parallel to

Af ; then from the Nature of the Curve, LM

(= Absciss An) will be the Logarithm of the

Ratio of the Ordinate AC to the Ordinate

Mn ; that is, of AC to AL : and consequent-

ly LM is the Logarithm of $\frac{AC}{CF - AF}$. So like-

wise lm is the Logarithm of $\frac{aP}{Pf - AF}$. And

supposing the invariable Subtangent AF , or

$af(b)$ equal to the Module of the Logarithms,

* Art. 14. the Line LM will be * $= AF \left| \frac{AF + CF}{AC} \right| =$

$AF \left| \frac{AC}{CF - AF} \right|$, and $lm = AF \left| \frac{AF + Pf}{aP} \right| =$

AF

$AF \left| \frac{aP}{Pf - Af} \right|$; therefore if to $Cf - Pf$, the Difference of the Tangents, you add $lm - LM$, the Difference of the Parallels; the whole will be equal to the Length CP of the Curve.

E X A M P L E VIII.

47. **T**HE Sine PM of any Arch PM of a Circle being given, together with the Radius: to find the Length of that Arch.

Let AP be $= x$, the Radius $AC = 1$, and the Sine or Ordinate $PM = y$. Then from FIG. 12.

the Nature of the Curve, $AP \times PB = \overline{PM}^2$, that is, $2x - xx = yy$. And throwing this Equation into Fluxions, we have $2\dot{x} - 2x\dot{x} = 2y\dot{y}$, and $\dot{x} = \frac{y\dot{y}}{1-x}$. Which squared will be

$$\dot{x}^2 = \frac{yy\dot{y}^2}{1-2x+xx} = \frac{y^2\dot{y}^2}{1-y^2}, \text{ because from the E-}$$

quation of the Curve $2x - xx = yy$; therefore

$$\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\frac{y^2\dot{y}^2}{1-y^2} + \dot{y}^2} = \sqrt{\frac{y^2\dot{y}^2 + \dot{y}^2 - y^2\dot{y}^2}{1-y^2}} =$$

$$\sqrt{\frac{\dot{y}^2}{1-y^2}} = \sqrt{\frac{\dot{y}}{1-y^2}} = \frac{1}{1-y^2} \dot{y}; \text{ which will be}$$

the Fluxion of the Arch AM . And the Flu-

$$\text{ent thereof will be } y + \frac{y^3}{2.3} + \frac{1.3.5}{2.4.5} y^5 +$$

$$\frac{1.3.5}{2.4.6.7} y^7 + \frac{1.3.5.7}{2.4.6.8.9} y^9, \&c. = \text{Length of the Arch } AM.$$

Now if the first Term be called A , the second B , the third C , the fourth D , &c. and the second Term be multiplied by $\frac{1}{2}$, the third by $\frac{3}{8}$, the fourth by $\frac{5}{16}$, &c. the Series aforesaid will

will become this $y + \frac{1}{2.3} Ay^3 + \frac{3}{4.5} By^5 +$
 $\frac{5}{6.7} Cy^7 + \frac{7}{8.9} Dy^9, \&c.$

Otherwise:

Draw the Radius MC , compleat the infinitely small Rectangle $PMnp$, make $AB=1$; and as before, $AP=x$, $PM=y$; then $x-xx=yy$, and because the small right-angled Triangle Mmn is similar to the Triangle PMC . Therefore $PM(\sqrt{x-xx}):MC(\frac{1}{2})::Pp$ or Mn

$$(\dot{x}):Mm \Rightarrow \frac{1}{2\sqrt{x-xx}} \dot{x} = \frac{\sqrt{x-xx}}{2x-2xx} \times \dot{x} = \text{Fluxi-}$$

on of the Arch AM ; and the Fluent thereof

$$\text{will be } x^{\frac{1}{2}} + \frac{1}{8}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}} + \frac{35}{1152}x^{\frac{9}{2}} \&c.$$

$$= x^{\frac{1}{2}} \text{ into } 1 + \frac{1}{8}x + \frac{3}{40}x^2 + \frac{5}{112}x^3 + \frac{35}{1152}x^4 \&c.$$

If the Cosine PC be made x , and the Radius AC be $=1$; then the Fluxion Mm of the Arch CD , being the Complement of AM to

a Quadrant, will be $\frac{1}{\sqrt{x-xx}} \dot{x}$ or $\frac{\sqrt{x-xx}}{x-xx} \dot{x}$; for

since the Nature of the Circle is $1-yy=xx$, this thrown into Fluxions is $-2y\dot{y}=2x\dot{x}$, viz.

$$y\dot{y}=x\dot{x}, \text{ and } \dot{y} = \frac{x\dot{x}}{y}, \text{ and } \dot{y}^2 = \frac{x^2\dot{x}^2}{yy} = \frac{x^2\dot{x}^2}{1-xx}$$

$$\text{and thence } \sqrt{x^2\dot{x}^2 + \dot{y}^2} = \sqrt{\frac{x^2\dot{x}^2}{1-xx} + \dot{x}^2} = \dot{x}$$

$\sqrt{\frac{1}{1-xx}}$. The Fluent of which will be $x + \frac{1}{6}$

$$x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7, \&c. = \text{Length of the Arch } CD.$$

SCHO-

SCHOLIUM I.

IF the Arch AM (suppose z) be given in the Series found above, and you want the versed Sine or Base AP (x), or the Sine MP ; then you must extract the Root of this Equation * $z = x^{\frac{1}{2}} + \frac{1}{8}x^{\frac{3}{2}} + \frac{1}{40}x^{\frac{5}{2}} + \frac{1}{112}x^{\frac{7}{2}}$, &c. AP * *Art. 3.* being $= x$, for the former; or of this, $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{1}{112}x^7$, &c. where $PM = x$.

SCHOLIUM II.

FROM this Example we have the Investigation of the Theorem of *Hugen's* for finding the Length of any Arch AM of a Circle by having the Chord AM thereof, and the Chord Am of half the same given. The Theorem is $\frac{8Am - AM}{3} = \text{Arch } AM$ nearly. FIG. 37.

Call the Radius AC , a , the Arch AM , z , the Chord AM thereof A , and the Chord Am of half the same B . Then will $A (=mp =$ twice the Sine of $Am = \frac{1}{2}z)$ be * $= z -$ * *Art. 3.*

$$\frac{z^3}{4.6aa} + \frac{z^5}{4.4.120a^4} - , \text{ \&c. and } B = \frac{1}{2}z -$$

$$\frac{z^3}{2.16.6aa} + \frac{z^5}{2.16.16.120a^4} - , \text{ \&c. Now mul-}$$

tiple B by any supposed Number n , and from the Product subtract A , and that the second Term

of the Remainder $-\frac{nz^3}{2.16.6a} + \frac{z^8}{4.6aa}$ may vanish,

make it $= 0$. Then there comes out $n = 8$; and so

$$8B - A = 3z * - \frac{3z^5}{64.120a^4} + , \text{ \&c. that is,}$$

$$\frac{8B - A}{3} = z; \text{ the Error being only } \frac{z^5}{7680a^4} -$$

\&c. too much.

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SCHOLIUM III.

FIG. 38. IF the Sagitta AP of any Arch MAm be continued out, and it be requir'd to find the Point F in the same, from which if the right Lines FME and Fme be drawn, they may include the Part Ee of the Tangent to the Circle in A very nearly equal to the Length of the Arch Mm .

Let C be the Centre, and $AG=a$ the Diameter, and the Saggita $AP=x$. Then will

$$PM (= \sqrt{ax - xx}) \text{ be } = a^{\frac{1}{2}} x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{8a^{\frac{3}{2}}} -$$

$$\frac{x^{\frac{7}{2}}}{16a^{\frac{5}{2}}} - \mathcal{E}c. \text{ and } AE (= \text{Arch } AM) = a^{\frac{1}{2}} x^{\frac{1}{2}}$$

$$+ \frac{x^{\frac{3}{2}}}{6a^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40a^{\frac{3}{2}}} + \frac{5x^{\frac{7}{2}}}{112a^{\frac{5}{2}}} + \mathcal{E}c. \text{ But because}$$

the Triangles FMP , FEA are similar, there-

$$\text{fore } AE - PM \left(\frac{1x^{\frac{3}{2}}}{4a^{\frac{1}{2}}} + \frac{1x^{\frac{5}{2}}}{12a^{\frac{3}{2}}} + \frac{3x^{\frac{7}{2}}}{64a^{\frac{5}{2}}} \mathcal{E}c. \right)$$

$$: AP (x) :: AE \left(a^{\frac{1}{2}} x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6a^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40a^{\frac{3}{2}}} + \frac{5x^{\frac{7}{2}}}{112a^{\frac{5}{2}}} \right.$$

$$\left. + \mathcal{E}c. \right) AF = \frac{2}{3}a - \frac{1}{3}x - \frac{12xx}{175a} - \text{or} + \mathcal{E}c.$$

Now let us suppose $AF = \frac{2}{3}a - \frac{1}{3}x$. Then if AH be taken $= \frac{1}{3}AP(x)$, and GF be taken equal to HC , a right Line drawn from F thro' M , and another thro' m , will cut off Ee nearly equal to the Length of the Arch MAm ; the Error being only $\frac{2 \times 16x^3}{525a^3} \sqrt{ax}$ or $-\mathcal{E}c.$

SCHO-

SCHOLIUM IV.

AND if the Area of any Segment MAm of FIG. 39. a Circle be wanted, nearly true, reduce the same to an infinite Series, *viz.* let the Seg-

$$\text{ment } MAm \text{ be} = \frac{4}{3} a^{\frac{\pi}{2}} x^{\frac{3}{2}} - \frac{2x^{\frac{5}{2}}}{5a^{\frac{1}{2}}} - \frac{x^{\frac{7}{2}}}{14a^{\frac{3}{2}}} -$$

$\frac{x^{\frac{9}{2}}}{36a^{\frac{5}{2}}}$, &c. to which $\frac{2}{3} \overline{AM} + \overline{PM} \times \frac{4}{3} \overline{AP}$ will be nearly equal, the Error being only $\frac{x^3}{70a^2} \sqrt{ax} +$ &c. too little.

EXAMPLE IX.

48. **T**O rectify the Ellipsis, or find the Length of any Arch AM thereof.

Let the Semi-transverse Axis be AC, a , and FIG. 40. the Semi-conjugate $Cc = b$; and let $AP = x$, $PM = y$. Now the Nature of the Curve will

be expressed thus $\frac{\overline{AP} \times \overline{Pa} \times \overline{Cc}}{\overline{AC}^2} = \overline{PM}^2$; that

is $\frac{bb}{aa} \times 2ax - xx = yy$. And throwing both

Sides of this Equation into Fluxions we shall have $2a\dot{x} - 2x\dot{x} = \frac{2a^2}{b^2} \times y\dot{y}$, or $a\dot{x} - x\dot{x} = \frac{a^2}{b^2} \times y\dot{y}$,

and so $\dot{x} = \frac{a^2}{b^2 - x^2} y\dot{y}$, and squaring both Sides $\dot{x}^2 =$

$\frac{a^4 y^2 \dot{y}^2}{a^2 b^4 - 2ab^2 x + b^4 x^2}$. And since from the Na-

ture of the Curve $bb \times 2ax - xx = aayy$, and $b^4 \times 2ax - xx = aabb yy$; if $aabb yy$ be put for its Equal in the Denominator, we shall have

$$\dot{x}^2 = \frac{a y^2 \dot{y}^2}{a^2 b^4 - a^2 b^2 y^2} = \frac{a^2 y^2 \dot{y}^2}{b^4 - b^2 y^2}, \text{ and adding}$$

$$\dot{y}^2 \text{ to both Sides; } \dot{x}^2 + \dot{y}^2 = \frac{a^2 y^2 \dot{y}^2}{b^4 - b^2 y^2} + \dot{y}^2 =$$

$$\frac{a^2 y^2 \dot{y}^2 + b^4 \dot{y}^2 - b^2 y^2 \dot{y}^2}{b^4 - b^2 y^2} = \frac{b^4 + a^2 - b^2 y^2}{b^4 - b^2 y^2} \times \dot{y} \dot{y};$$

$$\text{and consequently } \sqrt{x^2 + y^2} = \frac{\dot{y}}{b} \sqrt{\frac{b^4 + a^2 - b^2 y^2}{b^2 - y^2}}$$

= Fluxion Mm of the Arch AM of the Ellipsis. And supposing $b=1$, and $a^2 - b^2 (= a^2 - 1) = c$; then will Mm the Fluxion of the Arch be $y \sqrt{\frac{1 + cyy}{1 - yy}}$; and the Fluent of the

$$\text{same will be } y + \frac{ny^3}{3.2} + \frac{ny^5}{5.2} - \frac{n^2 y^5}{5.2.2.2} + \frac{ny^7}{7.2} - \frac{n^2 y^7}{7.2.2.2} + \frac{n^3 y^7}{7.2.2.2.2}, \text{ \&c. supposing } n=1+c.$$

But to find the Length of the Arch Mc of the Ellipsis, you may proceed thus: Here let PC be $= x$; then will $AC - PC \frac{\times Cc}{AC} = PM$

$$\text{that is, } \frac{bb}{aa - xx} \times \frac{bb}{aa} = yy; \text{ and so } \frac{bb}{aa - xx} \times \frac{bb}{aa} = aayy.$$

And throwing both Parts of the Equation into Fluxions, we shall have $-2bbxx = 2aayy$, or $-bbxx = aayy$, and $\dot{x} = \frac{aay\dot{y}}{-bbx}$,

$$\text{and squaring both Sides } \dot{x}^2 = \frac{a^4 y^2 \dot{y}^2}{b^4 x^2}.$$

Now from the Equation of the Curve we shall get $x^2 = aa - \frac{aa}{bb} y^2$, and so $b^4 x^2 = aab^4 - b^2 y^2$; and

so

so substituting $aab^4 - b^2y^2$ for b^4x^2 in the Denominator, and $\dot{x}^2 = \frac{a^4y^2\dot{y}^2}{aab^4 - a^2b^2y^2} = \frac{a^2y^2\dot{y}^2}{b^4 - b^2y^2}$

and then adding $\dot{y}^2$ to both Sides, there will be had $\dot{x}^2 + \dot{y}^2 = \left(\frac{a^2y^2\dot{y}^2}{b^4 - b^2y^2} + \dot{y}^2 \right) = \frac{b^4 + a^2y^2 - b^2y^2}{b^4 - b^2y^2}$

$\times \dot{y}^2 = \frac{b^4 + a^2 - b^2 \times yy}{b^4 - b^2y^2} \times \dot{y}^2$; therefore $\sqrt{\dot{x}^2 + \dot{y}^2}$

$= \sqrt{\frac{b^4 + a^2 - b^2yy}{b^4 - b^2yy}} \dot{y} = \text{Fluxion of the Arch}$

Mc. Which is the same as the Fluxion of the Arch *AM*, when *AP* was equal to *x*; and so the Fluent of this will be the same Expression as the Fluent of the Arch *AM*.

If *Aa* be $= a$, and *CP* $= x$, we shall have the Length of the Arch *Mc* =

$$\begin{aligned} & x \\ & + \frac{bb}{2aa} \times \frac{x^3}{3a} \\ & + \frac{b^2}{2aa} - \frac{b^4}{8a^4} \times \frac{x^5}{5a^4} \\ & + \frac{bb}{2aa} - \frac{b^4}{4a^4} + \frac{b^6}{16a^6} \times \frac{x^7}{7a^6} \\ & + \frac{bb}{2aa} - \frac{b^4}{8a^4} + \frac{3b^6}{16a^6} - \frac{5b^8}{128a^8} \times \frac{x^9}{9a^8}, \&c. \end{aligned}$$

for $\frac{bb}{aa} \times \frac{aa - xx}{aa - xx} = yy$; and $\frac{b}{a} \times \frac{aa - xx^{\frac{1}{2}}}{aa - xx^{\frac{1}{2}}} = y$.

Whence $\frac{b}{a} \times \frac{-x\dot{x}}{aa - xx^{\frac{1}{2}}} = \dot{y}$; and so $\frac{bb}{aa} \times \frac{xx}{aa - xx}$

$\times \dot{x} \dot{x} = \dot{y} \dot{y}$. Consequently $1 + \frac{bb}{aa} \times \frac{xx}{aa - xx}$

$\times \dot{x} \dot{x} = \dot{y} \dot{y} + \dot{x} \dot{x}$. And the Fluxion of the Arch

Arch Mc will be $1 + \frac{bb}{aa} \times \frac{xx}{aa-xx} \Big|^{1/2} \times \dot{x}$. Which
 thrown into an infinite Series will be $= 1 + \frac{1}{2}$
 $\times \frac{bb}{aa} \times \frac{xx}{aa-xx} - \frac{1}{8} \times \frac{b^4}{a^4} \times \frac{xx}{aa-xx} \Big|^2 \times \frac{1}{12} \times \frac{b^6}{a^6}$
 $\times \frac{xx}{aa-xx} \Big|^3 - \frac{5}{128} \times \frac{b^8}{a^8} \times \frac{xx}{aa-xx} \Big|^4 \&c. \times \dot{x} =$

$$+ \frac{1}{2} \dot{x} \times \frac{bb}{aa} \times \frac{x^2}{a^2} + \frac{x^4}{a^4} + \frac{x^6}{a^6} + \frac{x^8}{a^8}, \&c.$$

$$- \frac{1}{8} \dot{x} \times \frac{b^4}{a^4} \times \frac{x^4}{a^4} + \frac{2x^6}{a^6} + \frac{x^8}{a^8}, \&c.$$

$$+ \frac{1}{12} \dot{x} \times \frac{b^6}{a^6} \times \frac{x^6}{a^6} + \frac{3x^8}{a^8}, \&c.$$

$$- \frac{5}{128} \dot{x} \times \frac{b^8}{a^8} \times \frac{x^8}{a^8}, \&c.$$

And the Fluent of this will be

$$+ \frac{1}{2} \times \frac{bb}{aa} \times \frac{x^3}{3aa} + \frac{x^5}{5a^4} + \frac{x^7}{7a^6} + \frac{x^9}{9a^8}, \&c.$$

$$- \frac{1}{8} \times \frac{b^4}{a^4} \times \frac{x^5}{5a^4} + \frac{2x^7}{7a^6} + \frac{x^9}{9a^8}, \&c.$$

$$+ \frac{1}{12} \times \frac{b^6}{a^6} \times \frac{x^7}{7a^6} + \frac{3x^9}{9a^8}, \&c.$$

$$- \frac{5}{128} \times \frac{b^8}{a^8} \times \frac{x^9}{9a^8}, \&c.$$

being $=$ to the Series first proposed.

EXAMPLE X.

FIG. 41. 49. **T**O rectify the Hyperbola, or find the Length
 of any Arch AM thereof.

Let

APPENDIX.

III

Let C be the Centre, $aA(2a)$ the transverse Axis, $Cc(b)$ the Semi-conjugate, $AP(x)$ any Absciss, and $PM(y)$ the correspondent Ordinate. Now the Nature of this Curve is

$$\frac{aP \times AP \times \overline{Cc}^2}{aA^2} = \overline{PM}^2; \text{ that is, } \frac{bb}{aa} \times \overline{2ax + xx}$$

$= yy$. And throwing both Parts of the Equation into Fluxions we have $2a\dot{x} + 2x\dot{x} = \frac{2a^2}{b^2} \times y\dot{y}$, or $a\dot{x} + x\dot{x} = \frac{a^2}{b^2} \times y\dot{y}$; and so $\dot{x} =$

$$\frac{a^2 y \dot{y}}{b^2 x a + x^2}; \text{ and squaring both Sides } \dot{x}^2 =$$

$$\frac{a^4 y^2 \dot{y}^2}{a^2 b^4 + 2ab^4 x + b^4 x x}.$$

And since (from the Nature of the Curve) $bb \times \overline{2ax + xx} = aayy$, and $b^4 \times \overline{2ax + xx} = a^2 b^2 yy$; if $a^2 b^2 yy$ be put for its

$$\text{Equal } 2ab^4 x + b^4 x x, \text{ we shall have } \dot{x}^2 = \frac{a^4 y^2 \dot{y}^2}{a^2 b^4 + a^2 b^2 yy} = \frac{a^2 y^2 \dot{y}^2}{b^4 + b^2 yy}.$$

Then adding $\dot{y}^2$ to each Side, and $\dot{x}^2 + \dot{y}^2$ will be $= \frac{a^2 y^2 \dot{y}^2}{b^4 + b^2 yy}$

$$+ \dot{y}^2 = \frac{b^4 + a^2 + b^2 x yy}{b^4 + b^2 yy} \times \dot{y}^2. \text{ Consequently}$$

$$\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{y} \sqrt{b^4 + a^2 + b^2 x yy}}{b^4 + b^2 yy} = Mm, \text{ the}$$

Fluxion of the Arch AM ; being the very same Expression as that of the Ellipsis in the last Example, only with this Difference, that in the Ellipsis the Sign of the Term $b^2 yy$ is negative, and here it is affirmative. Consequently the Fluent here will be the same as the Fluent for the Ellipsis; only with the Alteration of the Signs, viz. here $a^2 + 1 = c$, and $c - 1 = n$.

Note,

Note, If the Hyperbola be an Equilateral one, then $2ax + xx = yy$ expresses the Nature of the same. This thrown into Fluxions will be $2ax + 2x\dot{x} = 2y\dot{y}$, or $a\dot{x} + x\dot{x} = y\dot{y}$, and $\dot{x} = \frac{y\dot{y}}{a+x}$. Which squared, and then $\dot{x}^2 =$

$$\frac{y^2 \dot{y}^2}{a^2 + 2ax + xx}; \text{ and substituting } yy (= 2ax + xx)$$

in the Denominator, $\dot{x}^2$ will be $= \frac{y^2 \dot{y}^2}{a^2 + yy}$.

Consequently $\sqrt{x^2 + y^2} = y \sqrt{\frac{a^2 + 2y}{a^2 + y^2}} =$ Fluxion of the Arch AM ; and making $a=1$, the same will be $y \sqrt{\frac{1+2y}{1+y^2}}$. The Fluent of

$$\text{which will be } y + \frac{y^3}{3.2} - \frac{y^5}{5.2} + \frac{y^7}{7.2.2} - \frac{y^9}{9.2.2.2}, \text{ \&c.}$$

SCHOLIUM.

FIG. 38. IF in the Ellipsis MAm , AG be $=$ transverse Axis, and $AQ =$ Latus Rectum, and QF be assumed $= \frac{1}{2} AQ \frac{19AG + 21AQ}{10AG} \times AP$; and in the Hyperbola, $QF = \frac{1}{2} AQ + \frac{19AC + 21AQ}{10AG} \times AP$, and the Secant FME be drawn; then the Tangent AE will be nearly equal to the Arch AM of the Ellipsis or Hyperbola, if the same be not very large.

EXAMPLE XVI.

FIG. 26. 50. TO rectify the Cycloid, or find the Length of any Arch AM thereof.

2

Let

Let $AQ=x$, $AB=1$; then $Qq=MS=\dot{x}$, and $PQ=\sqrt{x-xx}$ from the Nature of the Circle: therefore $AP=\sqrt{x}=x^{\frac{1}{2}}$. Consequently because the Triangles APQ , MmS are similar $AQ(x) : AP(x^{\frac{1}{2}}) :: MS(\dot{x}) : Mm = x^{-\frac{1}{2}}\dot{x}$ = Fluxion of the Arch AM ; the Flu-
ent of which will be $2x^{\frac{1}{2}}=2AP$ = Arch AM . So the Length AD of $\frac{1}{2}$ the Cycloid is = to twice the Diameter AB of the generating Circle, which we know to be true from other Principles.

EXAMPLE XVII.

51. **T**O find the Length of any Arch AM of the Cissoïd of Diocles AMI .

Let AB the Diameter of the generating Circle be a , and call PB, x , and the Ordinate PM, y ; then $AP=a-x$. Now from the Nature of the Curve $PM(y)$ is = $AP \times \sqrt{\frac{AP}{PB}}$ FIG. 42.

that is, $y = \overline{a-x} \sqrt{\frac{a-x}{x}} = x^{-\frac{1}{2}} \overline{x a-x}^{\frac{3}{2}}$.

And throwing this Equation into Fluxions, we

get $\dot{y} = \dot{x} \times -\frac{1}{2} x^{-\frac{3}{2}} \overline{x a-x}^{\frac{3}{2}} - \frac{3}{2} x^{-\frac{1}{2}} \overline{x a-x}^{\frac{1}{2}} \dot{x}$
 $= \frac{-a-2x}{2x} \sqrt{\frac{a-x}{x}} \times \dot{x}$. Which squar'd, and

$$\dot{y}^2 = \frac{aa+4ax+4xx \times \overline{a-x}}{4x^3} \times \dot{x}^2 =$$

$\frac{a^3+3aa x-4x^3}{4x^3} \times \dot{x}^2$. Whence $\dot{y}^2 + \dot{x}^2 =$

$\frac{a^3+3aa x}{4x^3} \times \dot{x}^2$, and so $\sqrt{\dot{y}^2 + \dot{x}^2} = \frac{a\dot{x}}{2x} \sqrt{\frac{a+3x}{x}}$

$= \frac{1}{2} a x^{-\frac{3}{2}} \dot{x} \times \sqrt{a+3x}$.

H h

Now

Now this must be compar'd with Mr. Cotes's 4th Form $d\dot{z}z^{\theta n + \frac{1}{2}n - 1} \sqrt{e + fz^n}$. And making $z = x$, $d = \frac{1}{2}a$, $\theta = -1$, $n = 1$, $e = a$, $f = 3$, we get $P(\sqrt{\frac{e + fz^n}{z^n}}) = \sqrt{\frac{a + 3x}{x}} \cdot R(\sqrt{f}) = \sqrt{3} \cdot T(\sqrt{\frac{e + fz^n}{z^n}}) = \sqrt{\frac{a + 3x}{x}}$, and $S(\sqrt{\frac{e}{z^n}}) = \sqrt{\frac{a}{x}}$. And so the Fluent $\frac{-2}{n} dP + \frac{2}{n} dR \left| \frac{R + T}{S} \right.$ becomes the Fluent of the given Flu-

$$\text{xion} = \sqrt{\frac{a + 3x}{x}} + a\sqrt{3} \left| \frac{\sqrt{3} + \sqrt{\frac{a + 3x}{x}}}{\sqrt{\frac{a}{x}}} \right. \text{ or } = \\ = a\sqrt{\frac{a + 3x}{x}} + 3\sqrt{\frac{1}{3}}aa \left| \frac{\sqrt{ax} + \sqrt{\frac{1}{3}}aa + 3ax}{\sqrt{\frac{1}{3}}aa} \right.$$

But this Fluent must be alter'd before it can express the Arch AM , because x begins at B and not at A .

In order to this, make $x = z$, and the said Fluent becomes $= -2a + 3\sqrt{\frac{1}{3}}aa \left| \frac{a + \sqrt{\frac{4}{3}}aa}{\sqrt{\frac{1}{3}}aa} \right.$; from which if the other Fluent be taken, we shall get $a\sqrt{\frac{a + 3x}{x}} = 2a +$

$$3\sqrt{\frac{1}{3}}aa \left| \frac{a + \sqrt{\frac{4}{3}}aa}{\sqrt{ax} + \sqrt{\frac{1}{3}}aa + ax} \right.$$

the Subtraction of the Ratio $\frac{\sqrt{ax} + \sqrt{\frac{1}{3}}aa + ax}{\sqrt{\frac{1}{3}}aa}$ from

the Ratio $\frac{\sqrt{a + \frac{4}{3}}aa}{\sqrt{\frac{1}{3}}aa}$ is the Division of the latter by the former) expressing the Length of the Arch AM .

Now

Now for the Construction of the Fluent.
 Draw $AC (\sqrt{\frac{4}{3}aa})$ cutting the Asymptote in C
 so, that CAB be $\frac{1}{3}$ of a right Angle; and let
 $BD (\sqrt{ax})$ be a mean Proportional between
 $AB(a)$ and $PB(x)$; and draw $CD (\sqrt{\frac{1}{3}1a+ax})$.
 Also draw AE bisecting PM , which will be
 $= \frac{1}{2}a\sqrt{\frac{a+3x}{x}}$. Then the Arch AP of the Cis-

$$\text{loid will be} = 2AE - 2AB + 3BC \left| \frac{AB+AC}{BD+DC} \right|$$

$$= a\sqrt{\frac{a+3x}{x}} - 2a + 3\sqrt{\frac{1}{3}aa} \left| \frac{a+\sqrt{\frac{4}{3}aa}}{\sqrt{ax}+\sqrt{\frac{1}{3}1a+ax}} \right|.$$





SECTION V.

Of the Use of Fluxions in the Cubation of Solids, and in the Quadrature of their Surfaces.

P R O B.

FIG. 7. 52. **T**O cube or find the Solidity of any Solid generated by the Rotation of a plain Figure AMN about the Axis AQ.

Draw the Ordinate pm infinitely near PM ; then the Parallelogram $PMrp$ may be taken for the Trapezium $PMmp$; and consequently the Cylinder described by the said little Parallelogram $PMrp$, while the Figure ANQ revolves about the Axis AQ , may be taken for the Increment or Fluxion of the Portion of the Solid, generated by the Rotation of the Portion AMP of the plain Figure; and the Fluent of that Fluxion will be equal to the Solidity of the said Portion, conceived as made up of an infinite Number of Cylinders of infinitely small equal Altitudes.

Now let $AP=x$, $PM=y$, and the Ratio of the Radius of a Circle to the Circumference, be expressed by $\frac{r}{p}$. Then will the Circumference of the Circle described by the Radius PM

$PMbe = \frac{py}{r}$, and the Area of the said Circle will be $\frac{p}{2r} y^2$. Which multiplied by $Pp(\dot{x})$ is

$\frac{p}{2r} \dot{x} y^2 =$ Solidity of the Cylinder aforesaid, or

Fluxion of the Portion of the Solid. And if for y^2 in this Expression you substitute its Value gained from the Equation of the Curve AMN , you will have a fluxionary Expression affected with only one unknown Quantity x ; and the Fluent thereof will express the Solidity required.

EXAMPLE I.

53. **T**O cube a right Cone.

A right Cone may be described by the Rotation of a right-angled Triangle ABC about the Side or Axis AB . Now let $AB = a$, $BC = r$, $AP = x$, $PM = y$. Then because the Triangles APM , ABC are similar, $AP(x) : PM(y) :: AB(a) : BC(r)$; therefore $\frac{rx}{a} = y$; and

squaring both Sides $\frac{r^2 x^2}{a^2} = y^2$. Therefore

$\frac{py^2 \dot{x}}{2r}$ (= Fluxion * of the Solid) = $\frac{pr^2 x^2 \dot{x}}{2a^2 r}$ * Art. 52.

= $\frac{prx^2 \dot{x}}{2a^2}$ (by substituting $\frac{rx}{a}$ for y) = the Fluxi-

on of the Solidity of the Part of the Cone generated by the Triangle APM . The Fluent of which will be = $\frac{prx^3}{6a^2} =$ Solidity of the

Part of the Cone; and if you substitute a for x ,

x , the Solidity of the whole Cone will be $\frac{pra^3}{6a^3} = \frac{1}{6} apr \times \frac{1}{3} a = \frac{1}{2} pr \times \frac{1}{3} a$; that is, the Base is to be multiplied into $\frac{1}{3}$ of the Altitude.

EXAMPLE II.

54. *TO cube a Sphere, or any Segment thereof.*

FIG. 44. A Sphere is generated by the Rotation of a Semicircle ABa about the Axis or Diameter ACa . Let the Radius $AC=r$, $AP=x$, $PM=y$; then from the Nature of the Circle generating it, $yy=2rx-xx$. Whence $\frac{py^2 \dot{x}}{2r} = \frac{px\dot{x} - px^2 \dot{x}}{2r} =$ Fluxion of the Segment of the Sphere generated by the Rotation of the Semi-segment of the Circle AMP ; and the Fluent thereof will be $\frac{\frac{1}{2}px^2 - px^3}{6r} =$ Solidity of the Segment AMm . And if the whole Diameter $2r$ be put for x , the Solidity of the whole Sphere will be $= \frac{2pr^2 - 8pr^3}{6r} = 2pr^2 - \frac{4}{3}pr^2 = \frac{2}{3}pr^2 = 2rp \times \frac{1}{3}r$; that is, the Rectangle under the Diameter $2r$, and Circumference p , is to be multiplied by $(\frac{1}{3}r)$ a third Part of the Radius, or sixth Part of the Diameter. And if the Diameter $2r$ be $= 1$, then the Solidity of the Sphere will be $\frac{1}{6}p$.

COROL. I.

HENCE a Sphere is equal to a Quadrangular Pyramid, whose Base is the Rectangle under the Diameter of the Sphere $2r$, and the Peri-

Periphery described by the same, and Altitude equal to the Semidiameter of the Sphere.

COROL. II.

AND because the Solidity of a Cylinder circumscribing the Sphere is pr^2 ; therefore it is to the Sphere as pr^2 to $\frac{2}{3}pr^2$, or as $3pr^2$ to $2pr^2$, or as 3 to 2.

EXAMPLE III.

55. IF *EDBF* be a right Cylinder, and the Part *DBMAF* be cut off from the same by a Plane *DFA* passing thro' *C* the Centre of the lower Base, and *F* the Extremity of the Diameter *FD* of the upper Base; it is required to cube the said Part *DBMAF*: which is called the Ungula or Hoof.

Let *AD* be at right Angles to *BE* = $2r$, FIG. 45. draw *CF*, any where in *AD* take the Point *P*, draw *PM* parallel to *CB*, and *PN* parallel to *CF*; join the Points *M*, *N*, and call the Altitude *FB*, *a*. Then any Triangle *PMN* (formed thus) right-angled at *M*, will be similar to the right-angled Triangle *CBF*. This being allow'd, call *CP*, *x*; then will $\overline{PM}^2 = rr - xx$, and $CB(r) : BF(a) :: PM(\sqrt{rr - xx}) : MN = \frac{a}{r}\sqrt{rr - xx}$; and the Area of the Triangle *PMN* will be $\frac{1}{2}PM \times MN = \frac{a}{2r} \times rr - \frac{axx}{2}$. Which multiplied by *Pp* ($\dot{x}$) will be $\frac{arr\dot{x} - axx\dot{x}}{2r}$ = the Fluxion of *APMN* the Part of the Ungula; and the Fluent will be $\frac{arx}{2} - \frac{ax^3}{6r}$ = Soli-

Solidity of the said Part $APMN$, and when $=x$ becomes $=r$, then will this last Expression be $\frac{1}{3}ar^3$, = Solidity of $\frac{1}{2}$ the *Ungula*, and so the Solidity of the whole will be $\frac{2}{3}ar^3$.

EXAMPLE IV.

FIG. 46. 56. **T**O cube a Solid generated by the Rotation of the Part FmM of the Lunule FAD of Hypocrates about the Radius ED as an Axis.

Draw the Tangent Aa . Call the Radius $CE, 1$; then the Radius CA will be $\sqrt{2}$. Also let $EP=x$, the Ordinate $PM=y$, and $Pm=z$.

Now $\overline{CA}^2 - \overline{CP}^2 = \overline{PM}^2$ from the Nature of the Circle; that is, $2 - 1 - 2x - xx (= 1 - 2x - xx) = yy$. And $\overline{ED}^2 - \overline{EP}^2 = \overline{Pm}^2$; that is, $1 - xx = zz$. Whence the Area of the Circle described by PM will be $\frac{p - 2px - pxx}{2r}$; and

the Area of the Circle described by Pm will be $\frac{p - pxx}{2r}$; and the Difference of these Areas

will be $\frac{px}{r} = \text{Anulus described by } Mm$. And

this drawn into the Fluxion $\dot{x}$ will be $\frac{p\dot{x}}{r} \dot{x} =$ to the Fluxion of the Part of the Solid described by the Part FmM of the *Lunule*; and the Fluent thereof will be $\frac{pxx}{2r} =$ Solidity of the said Part.

EXAMPLE-

EXAMPLE V.

57. **T**O cube a Parabolical Conoid generated by the Rotation of a Parabolick Space of any Kind about its Axis.

Let the Parameter be $= 1$; let AD be $=$ FIG. 1.
 a , $BD = r$, $AP = x$, $PM = y$. Then the Nature of all Parabola's will be expressed by $y^m = x$. Whence $y = x^{\frac{1}{m}}$; and so $yy = x^{\frac{2}{m}}$;

therefore $\frac{py^2 \dot{x}}{2r} = \frac{px^{\frac{2}{m}} \dot{x}}{2r} =$ Fluxion of the Solid

generated by the Rotation of the Portion APM of the Parabola. And the Fluent thereof will

be $\frac{mp}{2m+4xr} x^{\frac{m+2}{m}}$; and substituting $y^2 x$ for

$x^{\frac{m+2}{m}}$ its Equal, there will come out $\frac{mpy^2 x}{4+2mxr}$

$=$ Fluent of the said Solid.

And if a the Altitude of the whole Conoid be put for x , and $2r$ the Diameter of the Base for $2y$; then the Solidity of the whole Conoid

will be $\frac{mpr^2 a}{4+2mxr} = \frac{m}{4+2m} \times apr = \frac{1}{2} pr \times \frac{ma}{2+m}$.

Hence if the generating Parabola be the common one, m will be $= 2$; and so $\frac{m}{2+m}$ is

$= \frac{2}{2+2} = \frac{1}{2}$. Whence the Base is to be drawn

into $\frac{1}{2}$ the Altitude; and consequently the generated Conoid will be $\frac{1}{2}$ a Cylinder of the same Base and Altitude.

EXAMPLE VI.

58. **T**O cube a Spheroid generated by the Rotation of a Semi-elliptick Space about its transverse or conjugate Axes.

FIG. 40. — If you make $AP=x$, $PM=y$, and $Aa=a$, and the Parameter be $=b$; then will $\overline{PM}^2$ be $= AP \times b - \frac{\overline{AP}^2 \times b}{Aa}$; that is, $y^2 = bx - \frac{bx^2}{a}$

will express the Nature of the Ellipsis. Therefore * $\frac{py^2}{2r} \dot{x} = \frac{pbx}{2r} - \frac{pbx^2}{2ra} \times \dot{x} = \frac{pbx\dot{x}}{2r} - \frac{pbx^2\dot{x}}{2ra}$ will be the Fluxion of the Solid generated by the Rotation of the Part of the Ellipsis APM ; and the Fluent of the said Fluxion will be $\frac{pbx^2}{4r} - \frac{pbx^3}{6ar} =$ the said Solid.

And if the whole Axis a be substituted for x , the whole Spheroid will be $\frac{pba^2}{4r} - \frac{pba^2}{6r} = \frac{6pba^2 - 4pba^2}{24r} = \frac{pba^2}{12r}$.

Hence if the conjugate Axis Cc be $=2r$. Then will $4r^2$ be $=ab$; and so the Solidity of the whole Spheroid will become $\frac{1}{3}par$; that is, the Elliptick Spheroid is equal to a Cone of the same Height with a the transverse Axis of the Ellipsis, and Diameter of the Base equal to four times the conjugate Axis of the Ellipsis, viz. $4r$. And because the Altitude of a Cylinder circumscribing the Spheroid being a , and Diameter $=r$, the Solidity of that Cylinder is $\frac{1}{3}apr$; therefore a Spheroid, as well as a Sphere, is $\frac{2}{3}$ of the circumscribing Cylinder.

Other-

Otherwise:

This Example may be effected something shorter thus: Let $r=1$, in the general Expression $* \frac{py^2}{2r} \dot{x}$; then will the same become $\frac{py^2}{2} \dot{x}$. * Art. 52.

Make $Cc=1$, $AC=a$, and $PC=x$, and the Nature of the Ellipsis will be $1 - \frac{xx}{aa} = yy$;

and substituting $1 - \frac{xx}{aa}$ for yy in the said general

Expression; and $\frac{py^2 \dot{x}}{2}$ will be $= \frac{px}{2} - \frac{px^2 \dot{x}}{2aa}$

= Fluxion of the Part of the Spheroid generated by the Rotation of the Part $MPcC$ of the Ellipsis, and the Fluent thereof will be

$\frac{px}{2} - \frac{p}{6aa} x^3 = \frac{p}{2} x - \frac{x^3}{3aa}$ = Solidity of the said

Part of the Spheroid. And substituting a for

x , we shall have $\frac{p}{2} \times a - \frac{a}{3} = \frac{p}{2} \times \frac{3a - a}{3} =$

$\frac{2}{3} a \times \frac{p}{2}$; but $\frac{p}{2}$ = Circle described by the Ro-

tation of Cc ; therefore the Solidity of the Spheroid is $= \frac{2}{3}$ of a Cylinder of the same Base and Altitude.

EXAMPLE VII.

59. **T**O cube an Hyperbolical Conoid generated FIG. 41.
by the Rotation of a Semi-Hyperbola about the transverse Axis.

Let $AP=x$, $PM=y$, the Parameter $=b$, and the transverse Axis $=a$. Then the Nature

of the Curve will be $AP \times b + \frac{\overline{AP}^2 \times b}{Aa}$; that is, $y^2 = bx + \frac{bx^2}{a}$. Therefore $\frac{py^2 \dot{x}}{2r} = \frac{pbx \dot{x}}{2r} + \frac{pbx^2 \dot{x}}{2ra}$ is = Fluxion of the Conoid generated by the Rotation of the Part AMP of the Hyperbola about the Axis Aa ; the Fluent of which will be $\frac{pbx^2}{4r} + \frac{pbx^3}{6ra}$ = Solidity of the said Conoid. And when the Altitude of the Conoid is equal to a ; that is, when x is = a , the Solidity of the Conoid will become $\frac{6pba^2 + 4pba^2}{24r} = \frac{10pba^2}{24} = \frac{5}{12}pba^2$. And if $2r$ be = conjugate Axis, then will $2r = \sqrt{ab}$, and $4r^2 = ab$; which Value being substituted in the last Expression, and the Conoid will be $\frac{5}{12}par$.

If the Hyperbola be an Equilateral one, then the Nature of it will be expressed thus, $y^2 = ax + xx$; whence $\frac{py^2 \dot{x}}{2r} = \frac{apx \dot{x} + px^2 \dot{x}}{2r}$; and the Fluent thereof will be $\frac{apx^2}{4r} + \frac{px^3}{6r}$. And so since here $2r = a$, and $b = a$, the Solidity will be $\frac{5}{12}pa^2$.

Hence the Solidity of the circumscribing Cylinder will be $\frac{1}{2}par$; which therefore will be to the Solidity of the Conoid as $\frac{1}{2}par$ to $\frac{5}{12}par$, or as 3 to 10; and in the Conoid generated by the Equilateral Hyperbola, the circumscribing Cylinder is $\frac{1}{2}pa^2$. Whence the same will be to the Conoid as $\frac{1}{2}pa^2$ to $\frac{5}{12}pa^2$, or as 3 to 10.

By making $PC = x$, and $Cc = 1 = r$, and $AC = a$, as before in the Ellipsis; then the Nature of the Hyper-

Hyperbola will be $yy = \frac{xx}{aa} - 1$. Whence $\frac{py^2 \dot{x}}{2}$
 $= \frac{px^2 \dot{x}}{2aa} - \frac{px \dot{x}}{2}$; and the Fluent thereof will be
 $\frac{1px^3}{6aa} - \frac{px}{2}$. And since the Beginning of x is
 not at A , you must make $x=a$; and then this
 Fluent will become $= \frac{pa}{6} - \frac{pa}{2}$. Which being
 taken from the said Fluent, and we shall have
 $\frac{px^3}{6aa} - \frac{px}{2} + \frac{pa}{3} =$ Solidity of the Solid genera-
 ted by the Part AMP of the Hyperbola.

EXAMPLE VIII.

60. **T**O cube a Solid generated by the Rotation FIG. 47.
 of the interminate hyperbolical Space
 CABED about one of the Asymptotes CD.

Let $AB=a$, $AC=b$, $CP=x$, $PM=y$; and
 let $Pp=\dot{x}$; then let the Circumference of the
 Circle described by the Radius AC be $=p$, and
 the Circumference described by the Radius PC
 will be $\frac{px}{a}$. Which drawn into $PM(y)$ will give us

$\frac{pxy}{a}$, the Superficies of a Cylinder described by
 the Parallelogram $CPMR$. This again drawn
 into $Pp(\dot{x})$ will give us the Solidity of the
 little Cylinder $Pp qM$, viz. $\frac{pxy \dot{x}}{a} =$ Fluxion

of the Solid; but the Nature of the Hyper-
 bola with regard to the Asymptotes is $xy=ab$.

Whence $y = \frac{ab}{x}$, and $\frac{pxy \dot{x}}{a} = \frac{pabx \dot{x}}{ax} = pb \dot{x}$;
 the Fluent of which is $pbx =$ Solidity of the
 interminate Space $CABED$; and if a be put
 for x , the whole Solid will be pba .

EXAMPLE IX.

61. **T**O cube a Conoid generated by the Rotation of the hyperbolick Space AMBDC about
 FIG. 48. CD the half of the conjugate Axis of the Hyperbola AMB.

Call CA, a , CD, b , CP, x , PM, y , BD, r .
 Now $\frac{pr}{y}$ will be the Periphery described by

the Point M , and $\frac{pyy}{2r} =$ the whole Circle, having PM for a Radius; which multiplied by x , and $\frac{pyyx}{2r}$ will be the Fluxion of the So-

lid. But $yy = \frac{aaxx + aabb}{bb}$, from the Nature of the Curve; and substituting this Value in $\frac{pyyx}{2r}$, we have $\frac{aapxxx + aabbp\dot{x}}{2bbr}$ for the Flu-

xion of the Conoid, the Fluent of which is $= \frac{aapx^3}{6bbr} + \frac{aapx}{r}$; and substituting $\frac{bbyy}{xx + bb}$ for

aa , there arises $\frac{px^3yy}{6rxx + 6bbr} + \frac{bbpyyx}{2rxx + 2bbr} =$
 Solidity of the Conoid formed by the Space $AMPC$; and when x becomes $= b$, and y to r , then the whole Conoid will be $= \frac{bpr}{c}$.

COROL.

A Cylinder generated by the Rotation of the Parallelogram $ACSB$ about the Axis CS is $\frac{1}{2}pba$; and so it is to the said hyperbolick Solid

Solid as $\frac{1}{2} pba$ to pba , that is, as $\frac{1}{2}$ to 1, or 1 to 2.

EXAMPLE X.

62. *TO cube a Solid generated by the Rotation of a Parabola about a Semi-ordinate CB.*

Let $AB=r$, $BC=b$, $AP=x$, $PM=y$; then FIG. 49.
 if the Parameter be 1, this Equation will express the Nature of the Parabola $y^2=x$. But it is manifest *, that the Fluxion of the Solid * Art. 52.
 is the Circle described with the Radius MD drawn into $Dd=y$. Let the Ratio of the Radius to the Circumference be as r to p ; then $MD=BD=AB-AP=r-x$. And the Circumference of the Circle described by MD
 $=p-\frac{px}{r}$, and the Area of the said Circle will be $\frac{pr}{2}-px+\frac{px^2}{2r}$. Whence the Flu-

xion of the Solid will be $\frac{pr}{2}\dot{y}-px\dot{y}+\frac{px^2\dot{y}}{2r}$.

Now if in this Expression for x and x^2 you substitute y^2 and y^4 , their Equals (from the Equation of the Curve) we shall have $\frac{p\dot{y}}{2}-\frac{py^2\dot{y}}{2r}$

$+\frac{py^4\dot{y}}{2r}$ equal to the Fluxion of the indefinite

Part of the Solid generated by the Rotation of the Portion MCD about the Axis BC ;

the Fluent of which will be $\frac{1}{2}py-\frac{py^3}{3or}+$

$\frac{py^5}{10r}$ = said Part of the Solid.

But if for y^2 you put x in the general Expression *, then the said Solid will be $\frac{1}{2}py-$ * Art. 52.

$$\frac{pxy}{3or}$$

$\frac{pxy}{30r} + \frac{px^2y}{10r} = p \times \frac{1}{15}y - \frac{xy}{30r} + \frac{x^2y}{10r}$. And if for y you put b , and for x you put r , the whole Solid will be $p \times \frac{1}{15}br - \frac{1}{30}br + \frac{1}{10}br = 30 - 20 + 60 \times \frac{pbr}{60} = \frac{8}{15}pbr = \frac{1}{15}pr \times \frac{8}{15}b$; that is, the Base or Circle described by the Radius AB is to be drawn into $\frac{8}{15}$ of the Altitude BC .

Because a Cylinder of the same Base and Altitude is $\frac{1}{15}pbr$; therefore it will be to this parabolical Solid as $\frac{1}{15}pbr$ to $\frac{1}{15}pbr \times \frac{8}{15}b$; that is, as 1 to $\frac{8}{15}$, or as 15 to 8.

EXAMPLE XI.

FIG. 28. 63. *TO* cube a Solid generated from the Rotation of the Logarithmetical Curve about the Asymptote AH .

Here the Subtangent being always $= a, y\dot{x}$ is $= ay$; and so $\dot{x} = \frac{ay}{y}$. Whence $\frac{py^2\dot{x}}{2r} = \frac{payy}{4r}$ = Fluxion of Part of the Solid generated by the Rotation aforesaid; and the Fluent thereof will be $\frac{pay^2}{4r}$. And putting $r = AB$ for y , then will the whole Solid be $\frac{par^2}{4r} = \frac{1}{4}apr$.

Hence because a Cylinder, whose Altitude is $= a$, and Radius of the Base $= r$, is $\frac{1}{4}apr$: therefore it is to the Solid as $\frac{1}{4}a$ to $\frac{1}{4}a$, or as 2 to 1.

EXAMPLE XII.

FIG. 27. 64. *TO* cube a Solid generated by the Rotation of the Cissoïd about the Line AB as an Axis.

Let

Let $AB=1$, $AP=x$, $PM=y$; then the Nature of the Curve will be $y^2 = \frac{x^3}{1-x}$; and

so $\frac{py^2 \dot{x}}{2r} = \frac{px^3 \dot{x}}{2r \times 1-x}$; that is, (making $2r=AB=1$) $= \frac{px^3 \dot{x}}{1-x}$. The Fluent of which will be

$= \frac{1}{4}px^4 + \frac{1}{5}px^5 + \frac{1}{6}px^6 + \frac{1}{7}px^7$, &c. = Portion of the Solid described by APM . And putting $AB=1$, for x , we shall get $\frac{1}{4}p + \frac{1}{5}p + \frac{1}{6}p + \frac{1}{7}p$, &c. or $p \times \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$, &c. But this Series is infinite, as may be easily demonstrated from the Hyperbola; therefore the said Solid is infinite also.

Otherwise by the Measure of a Ratio.

If the Area APM of the Cissoïd of Diocles FIG. 50. revolves about the Base AB as an Axis, it is requir'd to cube the Solid generated thereby.

From the Nature of the Curve, calling AP, x , and PM, y , we have $x\sqrt{\frac{x}{a-x}} = y$. Whence

$\frac{pyy \dot{x}}{2r} = \frac{p}{2r} x \frac{x^3}{a-x} =$ Fluxion of the Solid requir'd.

Which compared with the first Form in Mr. Cotes's Tables, by making $d = \frac{p}{2r}$, $\theta = 4$,

$n=1$, $e=a$, $f=-1$; and we get $-\frac{px^3}{6r} - \frac{paxx}{4r}$

$-\frac{paax}{2r} - \frac{pa^3}{2r} \left| \frac{a-x}{a} \right.$ The Fluent of the given

Fluxion; or $-\frac{px^3}{6r} - \frac{paxx}{4r} - \frac{paax}{2r} + \frac{pa^3}{2r} \left| \frac{a}{a-x} \right.$

Since the Logarithm of the Ratio a to $a-x$ with an affirmative Sign, is the same as the

K k

Loga-

Logarithm of the Ratio $a-x$ to a with a negative Sign.

Now this Fluent may be thus constructed:

Let $AP(x)$, $AB(a)$, $AR\left(\frac{aa}{x}\right)$, $AS\left(\frac{a^3}{xx}\right)$

$AT\left(\frac{a^4}{x^3}\right)$ be continual Proportionals. Then

to the Module $TS\left(\frac{a^4-a^3x}{x}\right)$ assume PX equal to the Measure of the Ratio between $AB(a)$ and $PB(a-x)$; that is, make $PX = \frac{a^4-a^3x}{x} \bigg| \frac{a}{a-x}$; and from X towards B , lay off

XZ equal to $SR + \frac{RB}{2} + \frac{BQ}{3} = \frac{a^3-a^2x}{xx} + \frac{ax-ax}{2x} + \frac{a-x}{3}$; and the Solid will be equal to

a Cylinder, whose Base is PM , the Value of which is $\frac{px^3}{2r \times a-x}$; and Altitude PZ , the Value of which is

$-\frac{a^3-a^2x}{xx} - \frac{aa-ax}{2x} - \frac{a-x}{3}$

$+ \frac{a^4-a^3x}{x^3} \bigg| \frac{a}{a-x}$; therefore $\frac{p}{2r} \times \frac{x^3}{a-x} \times =$

$\frac{a^3-a^2x}{xx} - \frac{aa-ax}{2x} - \frac{a-x}{3} + \frac{a^4-a^3x}{x^3} \bigg| \frac{a}{a-x}$ is =

$-\frac{px^3}{6r} - \frac{paxx}{4r} - \frac{paax}{2r} + \frac{pa^3}{2r} \bigg| \frac{a}{a-x}$; the Flu-

ent to be constructed.

SCHOLIUM.

FIG. 50. 65. **T**HE Value of the infinite Solid generated by the Rotation of the Cissoïd AMI about the Asymptote BOH may be had thus:
Let

Let ANB be the generating Semicircle Circle, AP, x, AB, a ; then all the Ordinates PM do describe cylindrical Surfaces. Whence $r:p::$

$PB \times PM (x\sqrt{ax-xx})::\frac{px}{r}\sqrt{aa-xx} =$ Surface described by PM ; which multiplied by $\dot{x}$, and $\frac{p}{r}x\dot{x}\sqrt{ax-xx}$ is the Fluxion of the Solid generated as above.

Now to get the Fluent of that Fluxion, let us conceive the Generation Semicircle to revolve about an Axis parallel to the Asymptote BOH , and passing thro' the Point A ; then all the Ordinates PN will also describe cylindrical Surfaces; and so $\frac{px\dot{x}}{r}\sqrt{ax-xx}$ will be the Fluxion of the Solid generated by this Motion, which is equal to the Fluxion of the former Solid to be cubed. Whence the infinite Cissoidal Solid formed by the Revolution aforesaid, is equal to this latter Solid generated by the Revolution of the generating Semicircle about a right Line parallel to the Asymptote, and passing thro' the Point A .

PROP. II.

66. **T**O measure the Superficies of a Solid generated by the Rotation of a Figure FIG. 7.
 $AMNQ$ about its Axis AQ .

Let the Ratio of the Circumference to the Diameter of any Circle be $\frac{r}{p}$, let $AP=x, PM=y$; then will $Pp=Mq=\dot{x}, qm=\dot{y}$; and so $Mm=\sqrt{\dot{x}^2+\dot{y}^2}$.
the Circumference described by the Radius PM

$PM = \frac{py}{r}$. Which multiply'd by, or drawn into Mm , will be $= \frac{py}{r} \sqrt{\dot{x}^2 + \dot{y}^2} =$ Fluxion of Part of the Superficies of the Solid generated by the Rotation of the Part AMP of the Figure $AMN\mathcal{Q}$; and finding the Value of $\dot{x}^2$ from the Equation of the Curve thrown into Fluxions, and substituting the same in the general Expression; the Fluent thereof being afterwards found, will be the Superficies sought.

SCHOLIUM.

67. $\int \frac{p}{r} \sqrt{\dot{x}^2 + \dot{y}^2}$ be any Fluxion of a Superficies generated by the Rotation of a plain Figure about the Line x as an Axis; and if $\frac{p}{r}$ be the Ratio of the Radius to the Circumference of a Circle, and y represents any perpendicular Ordinate to the Absciss describing a Circle during the Generation of the Superficies; and if $\frac{p}{2r} z z$ be supposed the Fluent of that Fluxion, viz equal to a Circle whose Radius is z ; then will the Fluent of $2y\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$ be $= z z$, the Square of the Radius of the said Circle. Consequently if instead of finding the Fluent of the Fluxion $\frac{p}{r} y \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$ of any Superficies generated, as above, you find the Fluent of $2y\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$; this Fluent will be equal to the Square of the Radius of a Circle equal to the Fluent of the Fluxion $\frac{py}{r} \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, or to a Circle

cle

cle equal to the Superficies, whereof this last Expression is the Fluxion.

EXAMPLE I.

68. *TO find the Superficies of a right Cone.*

Because a right Cone is generated by the FIG. 43.

Rotation of the right-angled Triangle ABC about the Axis AB , the Value of $\dot{x}^2$ must first be gotten from the Equation of the Triangle thus: Let $AB=a$, $BC=r$, $AP=x$, $PM=y$. Now since $AP(x):PM(y)::AB(a):BC(r)$: therefore $rx=ay$. Which thrown into Flu-

xions will be $r\dot{x}=a\dot{y}$. Whence $\dot{x}=\frac{a\dot{y}}{r}$, and

$\dot{x}^2=\frac{a^2\dot{y}^2}{r^2}$; and substituting $\frac{a^2\dot{y}^2}{r^2}$ for $\dot{x}^2$ in the

general Expression $\frac{py}{r}\sqrt{\dot{x}^2+\dot{y}^2}$ we shall get

$$\frac{py}{r}\sqrt{\dot{x}^2+\dot{y}^2}=\frac{py}{r^2}\sqrt{a^2\dot{y}^2+r^2\dot{y}^2}=\frac{py}{r^2}\times\dot{y}\sqrt{a^2+r^2}$$

= Fluxion of Part of the Superficies of the Cone generated by the Triangle APM . The

Fluent of which will be $\frac{py^2}{2r^2}\sqrt{a^2+r^2}$ = said

Part of the Superficies; and substituting y for r , the Superficies of the whole Cone will be $=\frac{1}{2}p\sqrt{a^2+r^2}=\frac{1}{2}p\times AC$; that is, equal to the Rectangle under $\frac{1}{2}$ the Circumference of the Base into the Side AC of the Cone.

EXAMPLE II.

69. *TO find the Superficies of a Sphere, or of any Segment of it.* FIG. 45.

If

If the Diameter of the generating Circle be $=1$, then the Fluxion of the Arch Mm will be $= \frac{\dot{x}}{2\sqrt{x-xx}}$. Which drawn into the Circumference described by the Radius PM , will give $p\dot{x}$ the Fluxion of Part of the Superficies of the Sphere generated by the Semi-segment AMP ; and the Fluent px of it will be the Superficies of the Segment of the Sphere, having p for the Periphery of the circular Base, and x for the Height; and if you put the Diameter 1 for x , the Superficies of the whole Sphere will be equal to p , or (making $1=a$) $=ap$.

Whence any Segment of the Superficies of a Sphere is to the whole Superficies of the Sphere as px to p or x to 1; that is, as the Altitude of the Segment to the Diameter of the Sphere.

EXAMPLE III.

70. **T**O find the Superficies of a Parabolical Conoid.

The Equation expressing the Nature of the Parabola is $AP \times a = \overline{PM^2} (yy)$, or $ax = yy$. Which thrown into Fluxions, and $a\dot{x} = 2y\dot{y}$; whence $\dot{x} = \frac{4y\dot{y}}{a}$. Consequently $\frac{py}{r}\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{py}{ar} \times \sqrt{4y^2\dot{y}^2 + a^2\dot{y}^2} = \frac{py\dot{y}}{ar} \sqrt{4y^2 + a^2} =$ Fluxion of the Superficies of the Part of the Conoid generated by the Portion APM of the Parabola. Which comes under the first Case of the third Form in the little *Table of simple Curves that may be squared*, Art. 8. or under the

the third Form in the Tables of Mr. *Cotes*. For if in the little Table you make $d = \frac{p}{ar}$, $z = y$,

$n = 2$, $e = aa$, $f = 4$, we shall have the Fluxion $dzz^{n-1} \sqrt{e+fz} = \frac{pyy}{ar} \sqrt{4y^2+aa}$; and the Flu-

ent $\frac{2d}{3nf} R^3$ of it, by making $R (= \sqrt{e+fz}) =$

$\sqrt{4y^2+aa}$ will become $\frac{p}{12ar} \times 4y^2+aa^{\frac{3}{2}} =$

Fluent of the Fluxion of the Superficies of the Conoid. In like manner, in Mr. *Cotes*'s 3d Form,

the Fluent answerable to $\theta = 1$, viz. $\frac{2e+2fz^n}{3nf} dP$,

making $P (= \sqrt{e+fz^n}) = \sqrt{4y^2+aa}$; and substituting the same Values for d, n, e, f , as be-

fore, will become $\frac{2aa+8yy}{24} \frac{p}{ar} \sqrt{4y^2+aa} =$

$\frac{p}{12ar} 4y^2+aa^{\frac{3}{2}} =$ to that found by the little

Table of Quadratures.

This Fluent may be constructed thus: If r FIG. 51. be made $= y$, and you draw $PC = 2y$, and make $PB = a$, and join BC ; then will BC be $= \sqrt{4yy+aa}$. Call this n , and the Fluent to be

constructed will be $\frac{p}{12ay} n^3$. Let z be the Dia-

meter of the Circle equal to this Fluent; the Area of this will be $= \frac{pz^2}{8y}$. Therefore $\frac{p}{12ay} n^3$

must be $= \frac{pz^2}{8y}$; that is, $\frac{n^3}{12a} = \frac{z^2}{8}$; and so $\frac{n^3}{12a}$

$= z^2$. Whence $z = \sqrt{\frac{2}{3}} n^{\frac{2}{3}} = *$ Fluent of $*$ Art. 67.

$2y \sqrt{xx+yy}$.

Now

Now make $PH=BC$, and $PE=\frac{2}{3}BC$, join B, E . From H draw HF parallel to BE . Make $GP=PF$, bisect GH in D ; from which, as a Centre, describe the Semicircle GKH , cutting PM continued out in the Point K ; and then the right Line PK will be the Diameter of a Circle equal to the Superficies of the Conoid.

EXAMPLE IV.

FIG. 52, 53.

70. **T**O find the Superficies of a Spheroid generated by the Rotation of any Part AM of an Ellipsis about the Part DC of the Semi-axis BC .

Call the Semi-conjugate Axis AC, a , and the Semi-conjugate BC, b ; the Absciss PM, x , and the correspondent Semi-ordinate MP, y .

Now $\frac{bb}{aa} \times 2ax - xx = yy$, and $2ax - 2xx = \frac{2a^2}{b^2} \times yy$, or $ax - xx = \frac{a^2}{b^2} \times yy$. Whence $\dot{x} =$

$\frac{a^2}{b^2} \frac{y\dot{y}}{a-x}$, and $\dot{x}^2 = \frac{a^2 y^2 \dot{y}^2}{b^4 - b^2 y^2}$. Consequently the

Fluxion of the Arch $AM (= \sqrt{\dot{x}^2 + \dot{y}^2})$ will be $= \frac{\dot{y}}{b} \sqrt{\frac{b^4 + a^2 - b^2 yy}{b^2 - y^2}} = \frac{\dot{y}}{b} \sqrt{\frac{b^4 + ccyy}{b^2 - y^2}}$, (when $AC (a)$ is greater than $BC (b)$, and you put cc for $a^2 - b^2$) or $= \frac{\dot{y}}{b} \sqrt{\frac{b^4 - ccyy}{b^2 - y^2}}$ when $AC (a)$ is less than $BC (b)$, and cc be substituted for $b^2 - a^2$.

Again, DM is $= \frac{a}{b} \sqrt{bb - yy}$, from the Nature of the Curve. Whence the Periphery of the

the Circle described by DM will be $\frac{pa}{rb} \sqrt{bb-yy}$.

Which drawn into the Fluxion of the Arch before found, and there will arise

$$\frac{pay}{rbb} \sqrt{\frac{b^4 + ccyy \times bb - yy}{bb - yy}} = \frac{pay}{rbb} \sqrt{b^4 + ccyy}, \text{ when}$$

AC is greater than BC ; and $= \frac{pay}{rbb} \sqrt{b^4 - ccyy}$;

when AC is less than BC ; each being the Fluxion of the Superficies of a Spheroid, generated by the Rotation of the Part AM of the Ellipsis about the Part CD of the Axis; and may be referred to the fourth Form in the

Tables of Mr. Cotes. For making $d = \frac{pa}{rbb}$,

$z = y, n = 2, \theta = 0, e = b^4, f = \pm cc$, the Fluxion

$d z z^{\theta n + \frac{1}{2}n - 1} \sqrt{e + f z^n}$ will become

$\frac{pay}{rbb} \sqrt{b^4 \pm ccyy}$. And the Fluent $\frac{z^n}{n} dP + \frac{e}{nf}$

$dR \left| \frac{R+T}{S} \right|$, by making $P (= \sqrt{\frac{e+fz^n}{z^n}}) =$

$\frac{1}{y} \sqrt{b^4 \pm ccyy}$. $R (= \sqrt{f}) = c$, or $\sqrt{-cc}$,

$T (= \sqrt{\frac{e+fz^n}{z^n}}) = \frac{1}{y} \sqrt{b^4 \pm ccyy}$, $S (= \sqrt{\frac{e}{z^n}}) =$

$\frac{b^2}{y}$, will become $\frac{pay}{2rbb} \sqrt{b^4 \pm ccyy} + \frac{pab^2}{2rc}$

$\left| \frac{yc + \sqrt{b^4 \pm ccyy}}{b^2} \right|$. It being $\sqrt{b^4 + ccyy}$, and

the Measure of a Ratio, when $AC(a)$ is greater than $BC(b)$; and $\sqrt{b^4 - ccyy}$, and the Measure of an Angle, when $AC(a)$ is less than $BC(b)$.

FIG. 52.

Now to construct the Fluent in the first Case, we may proceed thus: Let F be one of the Foci. Make CF ($\sqrt{aa-bb} = \sqrt{cc} = c$, from the Nature of the Ellipsis) : CB (b) :: $CB(b) : CE = \frac{bb}{c}$, and draw the right Line

DE . Also make $CE\left(\frac{bb}{c}\right) : DE\left(\frac{1}{c}\sqrt{b^4+ccyy}\right)$

:: $CD(y) : KL = \frac{y}{bb}\sqrt{b^4+ccyy}$. To which

if you add LM , made equal to the Measure of the Ratio between $DE\left(\frac{1}{c}\sqrt{b^4+ccyy}\right) + DC$

(y) and $CE\left(\frac{bb}{c}\right)$ to the Module $CE\left(\frac{bb}{c}\right)$;

that is, if you assume $LM = \frac{b^2}{c} \left| \frac{cy + \sqrt{b^4+ccyy}}{b^2} \right|$,

then will $CA(a)$ be to $KM = KL + LM$

$\left(= \frac{y}{bb}\sqrt{b^4+ccyy} + \frac{b^2}{c} \left| \frac{cy + \sqrt{b^4+ccyy}}{b^2} \right| \right)$ as a

Circle described with $CA(a)$, as a Radius,

(which Circle is $= \frac{paa}{2r}$) is to the Superficies

of the Spheroid generated by the Rotation of

the Part AM of the Ellipsis $= \frac{pay}{2rb^2}\sqrt{b^4+ccyy}$

$+ \frac{pab^2}{2rc} \left| \frac{cy + \sqrt{b^4+ccyy}}{b^2} \right|$; which is the Fluent

to be constructed.

Now let the Radius of a Circle, equal to the Superficies of a Spheroid, be z . Then we shall have this Proportion, $CA(a) : KM ::$

Circle described by CA , viz. $\frac{paa}{2r} : \text{Superficies}$

of

of the Spheroid supposed equal to a Circle $\frac{pzz}{2r}$
 whose Radius is z . And multiplying the Ex-
 tremes and Means, there arises $\frac{pazz}{2r} = KM$.

$\times \frac{paa}{2r}$; and so $zz = KM \times a$; that is, a mean

Proportional between KM and $AC(a)$ will be
 the Radius of a Circle equal to the Superficies
 of the Spheroid generated by the Rotation of
 the Part AM of the Ellipsis.

When $DC(y)$ becomes equal to $BC(b)$,
 then will DE become BE ; and so if you

make $CE\left(\frac{bb}{c}\right) : BE\left(\frac{b}{c} \sqrt{bb+cc}\right) :: BC(b)$

$: KL = \sqrt{bb+cc} = AC = a$, and LM be =

Measure of the Ratio between $AC(a) + BC(b)$, and $CE\left(\frac{bb}{c}\right)$ to the Module $CE\left(\frac{bb}{c}\right)$,

we shall have CA to KM , as a Circle having
 CA for a Radius is to $\frac{1}{2}$ the Superficies of the
 whole Spheroid.

The Construction of the Fluent in the lat- FIG. 53
 ter Case is thus: In the Ordinate DM assume
 the Point E in such manner, that the Line CE

being drawn be equal to $\frac{bb}{c}$, or a third Pro-

portional to $CF(c)$ and $CB(b)$; and make CE

$\left(\frac{bb}{c}\right) : DE\left(\frac{1}{c} \sqrt{b^4 - ccyy}\right) :: DC(y) : KL =$

$\frac{y}{bb} \sqrt{b^4 - ccyy}$. Then if to KL you add LM

equal to the Measure of the Angle DCE to

the Module $CE\left(\frac{bb}{c}\right)$; $CA(a)$ will be to KM ,

as a Circle, whose Diameter is $CA (a)$, is to the Superficies of the Spheroid generated by the Rotation of the Part AM of the Ellipsis about the Part DC of the Axis; and so a mean Proportional between KM and CA , will be the Radius of a Circle equal to the said Superficies.

When $CD (y)$ becomes equal to $CB (b)$, draw BG perpendicular to CB . In which assume the Point e such, that Ce being drawn, be $= EC \left(\frac{bb}{c} \right)$. Then if you make CE or $Ce \left(\frac{bb}{c} \right) : Be \left(\frac{b}{c} \sqrt{bb - cc} \right) :: AC (b) : KL = \sqrt{bb - cc} = a$. And to the same you add LM equal to the Measure of the Angle eCB to the Module $eC \left(\frac{bb}{c} \right)$, we shall have CA to KM as a Circle having CA for a Radius is to $\frac{1}{2}$ the Superficies of the whole Spheroid.

EXAMPLE V.

71. **TO** find the Superficies of a Hyperbolical Conoid generated by the Rotation of any Part AM of an Hyperbola about the transverse Axis AP .

Let C be the Centre, Cc an Asymptote, $AC = b$ one of the Semi-axes, and $Ac = a$ the other. Let CP be $= y$, and $PM = x$. Now $\frac{aa}{bb} \times yy - bb = xx$, from the Nature of the Curve, and $2aayy = 2bbxx$; and so $\dot{x} = \frac{aay\dot{y}}{bbx}$. Consequently $\dot{x} \dot{x} = \frac{a^2yy\dot{y}\dot{y}}{b^4xx}$. And the Fluxion of the

the Arch AM ($= \dot{y}\dot{y} + \dot{x}\dot{x}$) will be $\frac{\dot{y}}{b} \sqrt{\frac{a^2yy + b^2yy - b^4}{yy - bb}} = \frac{\dot{y}}{b} \sqrt{\frac{ccyy - b^4}{yy - bb}}$, by putting

cc for $a^2 + b^2$. Again, PM is $= \frac{a}{b} \sqrt{yy - aa}$

from the Nature of the Curve; and therefore the Periphery described by PM will be

$\frac{pa}{rb} \sqrt{yy - aa}$. This drawn into the Fluxion of

the Arch before found, and there will arise

$\frac{pay}{rbb} \sqrt{\frac{ccyy - b^4 \times yy - bb}{yy - bb}} = \frac{pay}{rbb} \sqrt{ccyy - b^4}$, being

the Fluxion of the Superficies of an hyperbolic Conoid generated by the Rotation of the Part AM of the Hyperbola about the Axis AP . Which comes under the same Form in the Tables of Mr. Cotes, as the Fluxion of the Superficies of the Spheroid in the last Example.

And making $d = \frac{pa}{rbb}$, $z = y$, $\theta = 0$, $n = 2$,

$e = -b^4$, $f = cc$, $P = \frac{1}{y} \sqrt{ccyy - b^4}$, $R = c$,

$T = \frac{1}{y} \sqrt{ccyy - b^4}$, $S = \frac{b^2}{y}$, we shall have the

Fluent of that Fluxion thus expressed;

$\frac{pay}{2rbb} \sqrt{ccyy - b^4} - \frac{pab^2}{2rc} \left| \frac{cy + \sqrt{ccyy - b^4}}{b^2} \right|$. But be-

cause y begins at the Centre C , and not at A the Vertex, we must make $b = y$. Then the

first Part $\frac{y}{bb} \sqrt{ccyy - b^4}$ of the Fluent will be

$= \sqrt{cc - bb} = a$; which therefore must be taken from the said first Part. Moreover, the

Logarithmical Part $\frac{bb}{c} \left| \frac{y + \sqrt{ccyy - b^4}}{bb} \right|$, by ma-

king

king $b=y$, will become $\frac{bb}{c} \left| \frac{ab+bc}{bb} \right|$. Which

must be taken from $\frac{bb}{c} \left| \frac{cy + \sqrt{ccyy-b^4}}{bb} \right|$, to have

the true Logarithmical Part. Now because the Logarithm of the Ratio of bb to $ab+bc$ having an affirmative Sign, is the same as the Logarithm of the Ratio of $ab+bc$ to bb with a negative Sign (by *Def. 1. Sect. 2. Schol. 2.*)

therefore the Sum of $\frac{bb}{c} \left| \frac{cy + \sqrt{ccyy-b^4}}{bb} \right|$, and of

$\frac{bb}{c} \left| \frac{bb}{ab+bc} \right|$, will be equal to the Difference

sought. But the Ratio of $cy + \sqrt{ccyy-b^4}$ to bb , and the Ratio of bb to $ab+bc$, do compound the Ratio of $cy + \sqrt{ccyy-b^4}$ to $ab+bc$. Whence the Sum now mention'd will be

$\frac{bb}{c} \left| \frac{cy + \sqrt{ccyy-b^4}}{ab+bc} \right|$. Therefore the true Fluent

to be constructed will be $\frac{pay}{2rb^2} \sqrt{ccyy-b^4} - \frac{pa^2}{2r}$

$-\frac{pab^2}{2rc} \left| \frac{cy + \sqrt{ccyy-b^4}}{ab+bc} \right|$. And this may be done thus:

Let F be the Focus of the Hyperbola. Make CF ($\sqrt{aa+bb}=c$, (from the Nature of the Hyperbola): $CA(b) : : CA(b) : CE = \frac{bb}{c}$.

Draw EG perpendicular to CA meeting the Asymptote in G . In the Angle CEG , inscribe the right Line $CH=CP(y)$, which continue out to meet PM also continued out in the Point I . Then assume KL equal to $PI-Ac$

$= \frac{y}{bb} \sqrt{ccyy-b^4} - a$, since from the similar Tri-

angles CEH, CPI , we have $CE \left(\frac{bb}{c} \right) : EH$

$$\left(\frac{1}{c} \sqrt{ccyy - b^4} \right) :: CP(y) : PI = \frac{y}{bb} \sqrt{ccyy - b^4}.$$

Again, assume LM equal to the Measure of the Ratio between $CH(y) + EH$

$$\left(\frac{1}{c} \sqrt{ccyy - b^4} \right) \text{ and } CG(b) + EG \left(\frac{ab}{c} \right) \text{ to the}$$

Module $CE \left(\frac{bb}{c} \right)$. Then the Superficies generated by the Rotation of the Arch AM about the Axis AP , will be to a Circle described with the Semidiameter $Ac(a)$, viz. $\frac{paa}{2r}$,

as KM to Ac .

EXAMPLE VI.

72. **T**O find the Superficies of a hyperbolical Conoid generated by the Rotation of any Part AM of an Hyperbola about the conjugate Axis CPB .

Let C be the Centre, $CA=a$ the Semi-transverse Axis, $CB=b$ the Semi-conjugate, F the Focus, $PM=x$, any Ordinate to CA , and $CP=y$, the correspondent Abscifs. FIG. 55.

Now $\frac{bb}{aa} \times xx - aa = yy$, from the Nature

of the Curve. Whence $\frac{2bb}{aa} \times xx = 2yy$, that is,

$$\frac{bb}{aa} \times xx = yy, \text{ and } bbxx = aayy, \text{ and } x = \frac{aayy}{bbx}, \text{ and}$$

$$xx = \frac{a^2 yyy}{b^2 x}; \text{ and substituting } \frac{aayy + aabb}{bb} \text{ for } xx$$

$$\text{we shall get } xx = \frac{a^4 bbyyy}{aab^4 \times bb + yy} = \frac{a^2 yyy}{b^2 \times bb + yy}.$$

Whence

Whence $\sqrt{x\dot{x} + y\dot{y}} = \frac{y}{b} \sqrt{\frac{a^2 + b^2 \times yy + b^4}{bb + yy}}$. But

x is $= \frac{a}{b} \sqrt{bb + yy}$, which multiplied by $\frac{p}{r}$, and

$\frac{pa}{rb} \sqrt{bb + yy}$ will be the Periphery described by

the Point M ; and if the Fluxion of the Arch be drawn into this, we shall get

$$\frac{pay}{rbb} \sqrt{\frac{a^2 + b^2 \times yy + b^4 \times bb + yy}{bb + yy}} = \frac{pay}{rbb} \sqrt{a^2 + b^2 yy + b^4}.$$

And substituting cc for $a^2 + b^2$, the same will

become $\frac{pay}{rbb} \sqrt{ccyy + b^4}$. Which is the Fluxion

of the Superficies generated by the Rotation of the Arch AM about the Semi-conjugate Diameter CP .

Now this Fluxion being the same as that in the first Case of Example 4. aforegoing, the Fluent thereof must be the same as the Fluent

of that, viz. $\frac{pay}{2rbb} \sqrt{ccyy + b^4} + \frac{pab^2}{2rc} \left| \frac{y + \sqrt{cc + yyb^4}}{b^2} \right|$

Which may be thus constructed.

Make $CF (\sqrt{aa + bb} = c) : CB (b) :: CB (b) : CE = \frac{bb}{c}$, and draw the right Line PE .

$\left(\frac{1}{c} \sqrt{ccyy + b^4} \right)$ Then make $CE \left(\frac{bb}{c} \right) : PE$

$\left(\frac{1}{c} \sqrt{ccyy + b^4} \right) :: PC (y) : KL = \frac{y}{bb} \sqrt{ccyy + b^4}$.

And assume LM equal to the Measure of the

Ratio between $PE \left(\frac{1}{c} \sqrt{ccyy + b^4} \right) + PC (y)$;

that is, between $\frac{cy + \sqrt{ccyy + b^4}}{c}$, and CE

$$\left(\frac{bb}{c} \right)$$

$\left(\frac{bb}{c}\right)$ to the Module $CE \left(\frac{bb}{c}\right)$. And the Superficies generated by the Rotation of the Arch AM of the Hyperbola about the Semi-conjugate Axis CB , will be to a Circle described with the Semidiameter $CA (a)$, (which Circle is $\frac{paa}{2r}$) as the Sum of the Lines KL and LM to the said Semidiameter CA .

EXAMPLE VII.

73. **T**O find the Superficies generated by the Motion of the Arch CM of an equilateral Hyperbola about the Asymptote ABP .

Let A be the Centre, and let AB be equal FIG. 5. to BC , drawn parallel to the other Asymptote; assume AP , and draw PM parallel to the other Asymptote. Now call AB or BC , a , AP , x , and PM , y . Then from the Nature of the Curve, $aa = xy$. Whence $x = \frac{aa}{y}$, and

$\frac{aa}{x} = y$. Which thrown into Fluxions, and

$\dot{y} = \frac{-aa\dot{x}}{xx}$. Consequently $\dot{y}\dot{y} = \frac{a^4\dot{x}\dot{x}}{x^4}$. Again,

$x = \frac{aa}{y}$ thrown into Fluxions, and $\frac{aay}{yy} = \dot{x}$,

and $\frac{a^4\dot{y}\dot{y}}{y^4} = \dot{x}\dot{x}$. Whence $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} =$

$\sqrt{\frac{a^4\dot{x}\dot{x}}{x^4} + \frac{a^4\dot{y}\dot{y}}{y^4}} = \frac{\dot{y}}{yy} \sqrt{a^4 + y^4} = Mm$ the Flu-

vion of the Arch CM . Which drawn into

$\frac{py}{r}$, and the Product $\frac{p\dot{y}}{ry} \sqrt{a^4 + y^4} = \frac{p}{r} \dot{y} y^{-1} \sqrt{a^4 + y^4}$

M m

will

will be the Fluxion of the Superficies generated by the Rotation of the Arch CM as aforesaid. This Fluxion can be compared to that of the third Form of Mr. Cotes's Tables, and the Fluent will be had by making $d = \frac{p}{r}$, $z = y$, $\theta = 0$, $n = 4$, $e = a^4$, $f = y^4$, $P = \sqrt{a^4 + y^4}$.

$R = aa$, $T = \sqrt{a^4 + y^4}$, and $S = yy$: for then the same will be $\frac{1}{2} \frac{p}{r} \sqrt{a^4 + y^4} - \frac{1}{2} \frac{p}{r} aa$

$\left| \frac{aa + \sqrt{a^4 + y^4}}{yy} \right|$. But this Fluent must be corrected, because the Absciss AP (x) increases, while the correspondent Ordinate PM (y) decreases; therefore the Signs must be changed; that is, it will be $-\frac{1}{2} \frac{p}{r} \sqrt{a^4 + y^4} + \frac{1}{2} \frac{p}{r} aa$

$\left| \frac{aa + \sqrt{a^4 + y^4}}{yy} \right|$. Moreover, since the said Absciss AP (x) begins at A , and not at B , the said Fluent must undergo another Correction, as to Magnitude. And this is done thus: Make $y = a$, and then the first Part of the Fluent will become $\frac{1}{2} praa\sqrt{2}$. And this must

be added to $-\frac{1}{2} \frac{p}{r} \sqrt{a^4 + y^4}$, and the Sum will

be $\frac{1}{2} \frac{paa}{r} \sqrt{2} - \frac{1}{2} \frac{p}{r} \sqrt{a^4 + y^4}$, the true first Part of the Fluent. Again, the Logarithmick

Part $\frac{1}{2} \frac{p}{r} aa \left| \frac{aa + \sqrt{a^4 + y^4}}{yy} \right|$ must be alter'd;

which may be done by making $a = y$; for then it will become $\frac{1}{2} \frac{p}{r} aa \left| \frac{\sqrt{2} + 1}{1} \right|$. Which be-

ing

ing substracted from $\frac{1}{2} \frac{p}{r} a a \left| \frac{a a + \sqrt{a^4 + y^4}}{y y} \right|$,

and then we shall have $\frac{1}{2} \frac{p}{r} a a \left| \frac{\frac{a a}{y y} + \frac{\sqrt{a^4 + y^4}}{y y}}{\sqrt{2} + 1} \right|$,

which is the true Logarithmical Part. And consequently the true Fluent will be

$$\frac{p a a}{2 r} \sqrt{2} - \frac{p}{2 r} \sqrt{a^4 + y^4} + \frac{1}{2} \frac{p}{r} a a \left| \frac{\frac{a a}{y y} + \frac{\sqrt{a^4 + y^4}}{y y}}{\sqrt{2} + 1} \right|$$

which may be thus constructed.

Draw AM and AC , and from the Point C draw CG parallel to AM , meeting the Asymptote AP continued out in G . Now because of the similar Triangles APM , ABF , AP

$$\left(\frac{a a}{y y} \right) : PM(y) :: AB(a) : BF = \frac{y y}{a}.$$
 Whence

$$\frac{1}{a} \sqrt{a^4 + y^4} = AF.$$
 Again, because GC is parallel to AM , the Triangles APM , GBC are

$$\text{similar; therefore } PM(y) : AP \left(\frac{a a}{y} \right) :: CB(a)$$

$$: BG = \frac{a^3}{y y}.$$
 Whence $CG = \frac{a}{y y} \sqrt{a^4 + y^4}.$

Then if you make $AH = AC(a\sqrt{2}) - AF$

$$\left(\frac{1}{a} \sqrt{a^4 + y^4} \right) + \text{the Measure of the Ratio between } BG \left(\frac{a^3}{y y} \right) + GC \left(\frac{a}{y y} \sqrt{a^4 + y^4} \right), \text{ and}$$

$$AC(a\sqrt{2}) + AB(a) \text{ (which Ratio is =}$$

$$\frac{a a}{y y} + \frac{\sqrt{a^4 + y^4}}{y y} \left. \right) \text{ to the Module } AB(a); \text{ and}$$

the Superficies generated by the Rotation
M m z

tion of the Arch CM about the Asymptote AP , will be to a Circle whose Semidiameter is $AB(a)$, viz. $= \frac{paa}{2r}$, as $AH(a\sqrt{2} -$

$$\frac{1}{a}\sqrt{a^4+y^4}+a\left|\frac{\frac{a^3}{yy}+\frac{a\sqrt{a^4+y^4}}{yy}}{a\sqrt{2}+a}\right) \text{ to } BC(a);$$

and therefore the Value of AH drawn into $\frac{pa}{2r}$ will be equal to the said Superficies.

EXAMPLE VIII.

74. **T**O find the Superficies generated by the Rotation of the infinite Arch PZ of the Logarithmical Curve about its Asymptote AX .

FIG. 57. Let AP be an Ordinate at right Angles to the Asymptote, which call y ; let TP touch the Curve in P , and let the invariable Subtangent AT be $= a$. Draw pm parallel to AT , and infinitely near the Point P .

Then because the Triangles ATP, mpP are similar, $AP(y):TP(\sqrt{aa+yy})::mP(y):pP = \frac{y}{y}\sqrt{aa+yy} =$ Fluxion of the Arch PZ .

This multiplied by $2y$, and the Product $2y\sqrt{aa+yy}$ will be the Fluxion of a Square, whose Side is equal to the Diameter of a Circle equal to the Superficies sought by *Scho-tium*, Art. 67. which Square let $= AOq$.

Now this Fluxion may be *referr'd* to the fourth Form of Mr. Cotes's, by writing $y, 2, \theta, 2, aa, 1$, for z, n, θ, d, e, f . And again, writing for P, R, T, S , $\frac{1}{y}\sqrt{aa+yy}, 1, \frac{1}{y}\sqrt{aa+yy}$, and

$$\frac{a}{y},$$

$\frac{a}{y}$, we shall have the Fluent of the Fluxion

$$2y\sqrt{aa+yy} \text{ equal to } y\sqrt{aa+yy} + aa \left| \frac{y+\sqrt{aa+yy}}{a} \right|.$$

In order to construct this Fluent, we may observe that the right Line AO is a mean Pro-

portional between a and $\frac{y}{a}\sqrt{aa+yy} +$

$a \left| \frac{a+\sqrt{aa+yy}}{a} \right|$; and the Quantity $\frac{y}{a}\sqrt{aa+yy}$

is equal to the right Line EP at right Angles to the Curve in P , and bounded by the Asymptote in E . For because of the similar

Triangles TPA , APE , $TA(a):TP(\sqrt{aa+yy})$

$::AP(y):PE = \frac{y}{a}\sqrt{aa+yy}$. And the other

Quantity $a \left| \frac{y+\sqrt{aa+yy}}{a} \right|$ is the Measure of the

Ratio between $AP+TP$, and AT to the

Module AT . Or (because of the similar Tri-

angles APF , APE) the Measure of the Ra-

tio between $AE+EP$, and AP to the same

Module AT .

Hence the Fluent is thus constructed. Draw

$PE \left(\frac{y}{a}\sqrt{aa+yy} \right)$ perpendicular to the Curve

in P terminating at the Asymptote at E . Con-

tinue out the Ordinate AP to L , so that AL

be $= AE + EP \left(\frac{yy}{a} + \frac{y}{a}\sqrt{aa+yy} \right)$, and draw

LK the same way as the Tendency of the

Curve, viz. towards Z ; which make $= EP$

$\left(\frac{y}{a}\sqrt{aa+yy} \right)$; and let the same cut the Curve

in M . Lastly, between $KM \left(\frac{y}{a}\sqrt{aa+yy} + \right.$

$a \sqrt{\frac{y + \sqrt{aa + yy}}{a}}$ and AT , find a mean Proportional AO , (LM from the Nature of the Curve being the Logarithmick Part;) which will be the Semidiameter of a Circle equal to the Superficies generated by the Rotation of the infinite Arch PZ .

EXAMPLE IX.

75. **T**O find the Superficies generated by the Rotation of the infinite Arch MZ or AZ of the Cissoïd of Diocles about its Asymptote AG .

FIG. 42. Draw the Ordinate MP at right Angles to the Asymptote, and mp infinitely near to it. Then AB being $=a$, and $AN=x$, MN being at right Angles to AB , the Fluxion Mm of the Arch MZ will be $\frac{ax\sqrt{4a-3x}}{2 \times a - x\sqrt{a-x}}$;

which multiplied by $\frac{p}{r} \times MP = \frac{p}{r} \times \sqrt{a-x}$, and the Fluxion of the Superficies generated by the abovesaid Rotation of the Arch MZ will be had, viz. $\frac{pax}{2r} \sqrt{\frac{4a-3x}{a-x}}$.

Now this may be brought under the eleventh Form of Mr. Cotes's. For making $z=x$, $\theta=1$, $n=1$, $d=\frac{pa}{2r}$, $e=4a$, $f=-3$, $g=a$, and $b=-1$, the Fluxion in the eleventh Form $d\dot{z}z^{\theta n-1} \sqrt{\frac{e+fz^n}{g+bz^n}}$ will become $\frac{pax}{2r} \sqrt{\frac{4a-3x}{a-x}}$,

and the Fluent $\frac{1}{nb} dP \mathcal{Q} + \frac{eb-fg}{nfb} dR \left| \frac{R+T}{S} \right|$, by making $P(\sqrt{e+fz^n}) = \sqrt{4a-3x}$, $\mathcal{Q}(\sqrt{g+bz^n}) =$

$$= \sqrt{a-x}, R\left(\sqrt{\frac{f}{b}}\right) = \sqrt{3}, T\left(\sqrt{\frac{e+fz''}{g+bz''}}\right) =$$

$$\sqrt{\frac{4a-3x}{a-x}}, S\left(\sqrt{\frac{eb-fg}{b, g+bz''}}\right) = \sqrt{\frac{-a}{-a+x}} \text{ or } \sqrt{\frac{a}{a-x}},$$

will become $-\frac{pa}{2r} \times \sqrt{4a-3x} \times \sqrt{a-x} - \frac{pa}{2r}$

$$\times \frac{a}{\sqrt{3}} \left| \frac{\sqrt{3a-3x} + \sqrt{4a-3x}}{\sqrt{a}} \right|, \text{ being the Fluent}$$

of the given Fluxion $\frac{pax}{2r} \sqrt{\frac{4a-3x}{a-x}}$; wherein the negative Signs of the two Parts may be alter'd.

Now this Fluent may be constructed thus: Bisect MN in F , and draw AFE meeting the Asymptote in E . Make DB a mean Proportional between AB and NB . Also let the Angle $CAB = \frac{1}{3}$ of a right Angle, and continue out the Asymptote towards B meeting AC in the Point C . Then $BC = \frac{a}{\sqrt{3}}$ for $\sqrt{BC^2 + AB^2}$

$= 2BC^2$. Whence $BC^2 + AB^2 = 4BC^2$, and so $\frac{AB^2}{3} = BC^2$. Consequently $BC = \frac{AB}{\sqrt{3}} =$

$\frac{a}{\sqrt{3}}$. Again, $AF = \frac{x}{2} \sqrt{\frac{4a-3x}{a-x}}$. And because of the similar Triangles ANF , ABE , $AN(x) : AF\left(\frac{x}{2} \sqrt{\frac{4a-3x}{a-x}}\right) :: NB(a-x) :$

$FE = \frac{1}{2} \sqrt{4a-3x} \times \sqrt{a-x}$. And $DC = \sqrt{\frac{4aa-3ax}{3}}$. Now the Superficies generated

by the Rotation of the infinite Arch MZ , will be to a Circle $\left(= \frac{paa}{2r}\right)$ whose Semidiameter is

I

AB

$AB (a)$, as $2 EF (\sqrt{4a-3x} \times \sqrt{a-x}) +$ the Measure of the Ratio between $BD (\sqrt{aa-ax}) + DC (\sqrt{\frac{4aa-3ax}{3}})$, and $BC (\frac{a}{\sqrt{3}})$ to the Module $BC (\frac{a}{\sqrt{3}})$.

Now making $x=0$ in the aforesaid Fluent, and it will become $\frac{pa}{2r} \times 2a +$

$\frac{pa}{2r} \times \frac{a}{\sqrt{3}} \left| \frac{\sqrt{3a} + 2\sqrt{a}}{\sqrt{a}} \right|$. Which is the Fluent expressing the Quantity of the infinite Arch AZ . Being to a Circle whose Semidiameter is $AB (a)$, as $2 AB +$ the Measure of the Ratio between $BA (a) + AC (\frac{2a}{\sqrt{3}})$ and $BC (\frac{a}{\sqrt{3}})$ to the Module BC . And the Difference of these Fluents, omitting $\frac{pa}{2r}$, viz. $2 AB (2a) - EF (\sqrt{4a-3x} \times \sqrt{a-x}) +$ Measure of the Ratio $BA (a) + AC (\frac{2a}{\sqrt{3}})$ to $BD (\sqrt{aa-ax}) + DC (\sqrt{\frac{4aa-3ax}{3}})$, the Module being $BC (\frac{a}{\sqrt{3}})$, will give us the Quantity of the finite Arch AM . For it will be to a Circle having BA for a Semidiameter, as that Difference is to the same Semidiameter BA .

When the Curve revolves about the Base AB , Mr. Cotes in *Harmonia Mensurarum* gives the Quantity of the Superficies generated. But the Operation is long and troublesome, on account of the Fluxion's not directly coming under any of his Forms.



S E C T I O N VI.

Of the Use of Fluxions in finding the Centres of Gravity of Figures.

P R O B.

76. **T**O find the Centre of Gravity of a plain Figure.

Let AB be the Axis, and MN an Ordinate FIG. 58.
to it; and let mn be another infinitely near MN , and suppose C to be the Centre of Gravity.

Now if the infinitely small Parts, as $MmnN$ of the Figure be conceived as so many Weights hung on the Axis AB , at the several Points P, P, P , &c. and the Point of Suspension be in A the Vertex of the Figure; the Distance of the Centre of Gravity C from A the Point of Suspension, will be equal to the Quotient of the Division of the Sum of the *Momentums* of all the said infinitely small Parts or Weights $MmnN$ by the Sum of all those little Parts or Weights; that is, by the Area of the whole Figure. This is plain from common Books of Mechanicks.

Therefore calling $AP, x, MP, y, Pp, \dot{x}$, the infinitely small Part or Weight $MmnN$ will be $= 2y\dot{x}$, and the Sum of them $=$ Fluents of $2y\dot{x}$. And the *Momentum* of one of those infinitely

N n
finitely

finitely small Parts or Weights will be $2y\dot{x}$ multiplied by $AP(x) = 2yx\dot{x}$. The Fluent of which divided by the Fluent of $2y\dot{x}$ will be $= AC$, the Distance of the Centre of Gravity from A , the Vertex of the Figure.

EXAMPLE I.

77. **T**O find the Centre of Gravity of a Triangle ADE .

FIG. 59.

Draw the right Line AB bisecting the Base DE in B . Then because the Triangle $ABD = ABE$, each of them may be resolved into an infinite Number of little Parts or Weights $MmpP$, $PNnp$, each equal to one another on either Side the Line AB taken as an Axis; and consequently the Centre of Gravity C must be somewhere in the said Line AB .

Now call AB, a , DE, b ; AP, x , MN, y ; and draw AF perpendicular to DE , which call c . Then since the Triangles AMN, ADE are similar, $AB(a) : DE(b) :: AP(x) : MN = \frac{bx}{a}$. Also because the Triangles APQ, ABF are similar; therefore $AB(a) : AF(c) :: AP(x) : AQ = \frac{cx}{a}$. And as $AP(x) : AQ\left(\frac{cx}{a}\right) ::$

$Pp(\dot{x}) : Qq = \frac{c\dot{x}}{a}$. Whence the Momentum

$yx\dot{x}$ will be $= \frac{cbx^2\dot{x}}{a^2}$; the Fluent of which is

$= \frac{cbx^3}{3a^2}$. Which being divided by $\frac{cbx^2}{2a^2}$, the A-

rea of the Triangle ADE , and the Quotient will be $\frac{2}{3}x =$ Distance of the Centre of Gravity

vity of the Part AMN of the Triangle ADE from the Vertex; and substituting a for x , the Distance of the Centre of Gravity of the whole Triangle ADE from the Vertex A will be $= \frac{2}{3} AB(a)$.

E X A M P L E II.

78. **T**O determine the Centre of Gravity C in the common Parabola.

Calling AP, x , MN, y , and AB, a , and the Parameter p ; then will px be $= yy$; and so $y = \sqrt{px}$. Whence $y \dot{x} = \dot{x} \sqrt{px}$; and so the Momentum $y \dot{x} = \dot{x} \sqrt{px} = \dot{x} p^{\frac{1}{2}} x^{\frac{1}{2}}$. The Fluent of which is $\frac{2}{5} p^{\frac{1}{2}} x^{\frac{5}{2}}$; but the Fluent of $y \dot{x}$ is $= \frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}}$ = Area of the Portion AMN of the Parabola; therefore dividing $\frac{2}{5} p^{\frac{1}{2}} x^{\frac{5}{2}}$ by $\frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}}$, and the Quotient will be $= \frac{3}{5} x = AC$. And substituting a for x , the Distance of the Centre of Gravity C from the Vertex A will be $= \frac{3}{5} AB$.

E X A M P L E III.

79. **T**O determine the Centre of Gravity C in all Parabola's of any higher Kind.

Here $p^n x^m = y^r$ expresses the Nature of all Curves of this Kind. Whence $y = p^{\frac{n}{r}} x^{\frac{m}{r}}$, and consequently $y \dot{x} = p^{\frac{n}{r}} x^{\frac{m}{r}} \dot{x}$, and the Momentum

N n 2

$x y \dot{x}$

$xy\dot{x} = p^{\frac{n}{r}} x^{\frac{m+r}{r}} \dot{x}$. The Fluent of which is $=$
 $\frac{r}{m+2n} p^{\frac{n}{r}} x^{\frac{m+2r}{r}}$. Which divided by the Flu-

ent of $y\dot{x} = \frac{r}{m+n} p^{\frac{n}{r}} x^{\frac{m+n}{r}}$, and the Quoti-

ent will be $= \frac{m+r}{m+2r} x =$ the Distance of the

Centre of Gravity of the Portion MAN from the Vertex A ; and substituting a for x , there

comes out $\frac{m+r}{m+2r} a = AC$.

EXAMPLE IV.

FIG. 60. 80. **T**O find the Centre of Gravity of the Space ADE contained under two equal Parabola's AD , AE , touching one another in their Vertex A , and the straight Line DE parallel to the common Axis of the Parabola's.

Make $AP = x$, $PM = y$, and let the Parameter be $= 1$: then will $\overline{AP}^2 (x^2)$ be $(PM \times 1) = y$; therefore the Momentum $xy\dot{x} = x^3\dot{x}$.

The Fluent of which is $\frac{1}{4}x^4$; but the Fluent

of $y\dot{x}$ is $\frac{1}{3}x^3$. Whence $\frac{\frac{1}{4}x^4}{\frac{1}{3}x^3} = \frac{3}{4}x = AC$, the Distance of the Centre of Gravity from A .

If the Parabola's AME , AMD be of any Kind whatsoever, this Equation will express the Relation of AP to PM , viz. $x^m = y^n$.

Therefore $y\dot{x} = x^{\frac{m}{n}}\dot{x}$, and the Momentum $xy\dot{x}$
 $= x^{\frac{m+n}{n}}\dot{x}$. The Fluent of which is $\frac{nx}{m+2n}$

but

but the Fluent of $y \dot{x}$ is $\frac{n}{m+n} x^{\frac{m+n}{n}}$. Whence the Quotient of the Division of the former of these Expressions by the latter, viz. $\frac{m+n}{m+2n} x$ will be $= AC$.

EXAMPLE V.

81. **T**O determine the Centre of Gravity of any Arch of a Circle.

Let BE the Chord of the given Arch BDE FIG. 61. be parallel to the Diameter FG ; which being consider'd as an Axis, to which the small Weights BM are suspended; and so the *Momentums* of them as $BM \times PB$. And since the Numbers and *Momentums* of the said little Weights on each Side the Radius AD bisecting the Arch BDE are equal, the Centre of Gravity will be in AD .

Now let $AB = a$, $AP = HB = x$, and $Pp = Bm = \dot{x}$. Then $PB (\sqrt{aa - xx}) : AB (a) :: Bm (\dot{x}) : BM = \frac{a \dot{x}}{\sqrt{aa - xx}}$, and $BM \times BP = \sqrt{aa - xx} \times \frac{a \dot{x}}{\sqrt{aa - xx}} = a \dot{x} = AB \times BM$;

and the Sum of the *Momentums*, or Fluent of this Fluxion, is $a x = AB \times BH$. Which divided by the Arch BD and $\frac{AB \times BH}{BD}$ is = the

Distance of the Centre of Gravity C from the Centre A of the Circle. And substituting the Quadrant FD for BD , and the Radius FH or AB for BH ; then will $\frac{AB^2}{FD}$ be the Distance

of

of the Centre of Gravity C of the Semicircle FDG from the Centre A .

EXAMPLE VI.

82. **T**O determine the Centre of Gravity of a Sector of a Circle ABE .

FIG 62. The Centre of Gravity will be in the Radius AD , which bisects the Arch BE . Describe the Arch MPM with any Distance AP , and another Arch mpm infinitely near it; then the *Momentum* of the Arch MPM drawn into Mm or Pp , will be the *Momentum* of the annular Segment $mMPMm$, or the Fluxion of the *Momentum* of the Sector.

Now let AB be $=a$, $AF=b$, and $BD=c$; then the *Momentum* of the Arch BDE ($2ab$) is to the *Momentum* of the Arch MPM , as the Triangle ABE to the Triangle AMM , or as $\overline{AB}^2$ to $\overline{AM}^2$; since the Triangles ABE , AMM are similar. Whence the *Momentum* of the Arch $MPM = \frac{2abx^2}{a^2} = \frac{2bx^2}{a}$, and the *Momentum* of the annular Segment $mMPMm = \frac{2bxx}{a}$. The Fluents (or Sum) of which $\frac{2bx^3}{3a}$ divided by the Sum of the Weights, or the Area ac of the Sector, and the Quotient $\frac{2bx^3}{3aac}$ is the Distance of the Centre of Gravity of the Sector of the Circle MPM .

EXAMPLE VII.

83. **T**O find the Centre of Gravity of any Segment MAm of an Hyperbola.

Making

Making AC, a , CB, b , AP, x , and PM, y , FIG. 63.
we have from the Nature of the Curve

$$\frac{bb}{aa} \times 2ax + xx = yy. \text{ Therefore the Fluxion of}$$

the *Momentums* will be $\frac{2b}{a} x^{\frac{3}{2}} \dot{x} \sqrt{2a+x}$, and

the Fluxion of the *Weights* $\frac{2b}{a} x^{\frac{1}{2}} \dot{x} \sqrt{2a+x}$.

Each of which may be compared with that of the fourth Form in the Tables of Mr. *Cotes*; and so making $\theta = 2$, $n = 1$, &c. we shall have the Sum of the *Momentums* =

$$\frac{-3aa + ax + 2xx}{3} \times y + 2a^2b \left| \frac{\sqrt{x} + \sqrt{2a+x}}{\sqrt{2a}} \right|.$$

And again, putting $\theta = 1$, &c. the Sum of the

$$\text{Weights will be } = \overline{a+x} \times y + 2ab \left| \frac{\sqrt{x} + \sqrt{2a+x}}{\sqrt{2a}} \right|;$$

and so dividing the former Fluent by the latter, we have

$$\frac{-3aa + ax + 2xx \times y + 2a^2b \left| \frac{\sqrt{x} + \sqrt{2a+x}}{\sqrt{2a}} \right|}{\overline{a+x} \times y + 2ab \left| \frac{\sqrt{x} + \sqrt{2a+x}}{\sqrt{2a}} \right|}$$

$$= \frac{\overline{a+x} \times y + 2ab \left| \frac{\sqrt{x} + \sqrt{2a+x}}{\sqrt{2a}} \right|}{\overline{a+x} \times y + 2ab \left| \frac{\sqrt{x} + \sqrt{2a+x}}{\sqrt{2a}} \right|}$$

= Distance of the Centre of Gravity from the Vertex D .

Now to construct this Expression, make $CB(b) : CA(a) :: PM(y) : CF = \frac{ay}{b}$. And a-

gain, $PM(y) : CB(b) :: CA(a) : CG = \frac{ab}{y}$.

Then take CH equal to the Measure of the Ratio between $CA(a)$ and $FP(a+x - \frac{ay}{b})$.

Which Ratio is = duplicate Ratio of $\sqrt{2a}$
2 to

to $\sqrt{x - \sqrt{2a + x}}$. When this is done, if you

make as $3PH(3a + 3x - \frac{3ab}{y} \left| \frac{a}{a + x - \frac{ay}{b}} \right|)$

$: 2CF\left(\frac{2ay}{b}\right) :: CF\left(\frac{ay}{b}\right) : CZ =$

$$\frac{4ax + 2xx}{3a + 3x - \frac{3ab}{y} \left| \frac{a}{a + x - \frac{ay}{b}} \right|}.$$

This fourth Proportional will be = Distance of the Centre of Gravity z from the Centre C . For if $CA(a)$ be taken from it, we shall

$$\text{have } \frac{4ax + 2xx}{3a + 3x - \frac{3ab}{y} \left| \frac{a}{a + x - \frac{ay}{b}} \right|} - a =$$

$$\frac{-3aa + ax + 2xx \times y + 2a^2b \left| \frac{\sqrt{x + \sqrt{2a + x}}}{\sqrt{2a}} \right|}{a + x \times y + 2ab \left| \frac{\sqrt{x + \sqrt{2a + x}}}{\sqrt{2a}} \right|}$$

the Expression first found.

SCHOLIUM.

84. **T**HE Distance of the Centre of Gravity of the Segment of an Ellipsis or Circle from the Vertex will be expressed thus:

$$\frac{-3aa - ax + 2xx \times y - 2a^2b \left| \frac{\sqrt{x + \sqrt{2a - x}}}{\sqrt{2a}} \right|}{a - x \times y - 2ab \left| \frac{\sqrt{x + \sqrt{2a - x}}}{\sqrt{2a}} \right|}$$

it being the same as that for the Centre of Gravity

Gravity of the Hyperbola only with the Alteration of some Signs, and the Measure of the Ratio there being here changed into that of an Angle, because $R (= \sqrt{-f}) = \sqrt{-1}$ is negative.

EXAMPLE VIII.

85. **T**O find the Centre of Gravity of the external hyperbolick Space $AMmaA$. FIG. 64.

Let the Semi-conjugate Diameter $BC = b$, the Semi-transverse $AC = a$, the Absciss $CP = y$, and the Semi-ordinate $PM = x$.

Then from the Nature of the Curve,

$$\frac{b}{a} \sqrt{xx - aa} = yy; \text{ and so } x = \frac{a}{b} \sqrt{bb + yy}.$$

Whence $\frac{2a}{b} y \sqrt{bb + yy}$ is the Fluxion of the

Weights, and $\frac{2a}{b} y y \sqrt{bb + yy}$ that of the *Moments*.

And from the Comparifon of this Fluxion with the 3d Form of Mr. Cotes's Tables, θ being $= 1, \eta = 2, \&c.$ the following Fluent will be had,

viz. $\frac{2a}{3b} | bb + yy^{\frac{3}{2}}$; and making $y = 0$, the same will become $2ab^{\frac{3}{2}}$; which muft be taken from

that Fluent, and the Remainder $\frac{2a}{3b} | bb + yy^{\frac{3}{2}} - 2ab^{\frac{3}{2}}$

will be the Fluent of the Fluxion $\frac{2a}{b} y y \sqrt{bb + yy}$.

Alfo the Fluent of $\frac{2a}{b} y \sqrt{bb + yy}$, by comparing it with the fourth Form, θ being $= 0$,

$\eta = 2, \&c.$ will be $\frac{ay}{b} \sqrt{bb + yy} + ab \left| \frac{y + \sqrt{bb + yy}}{b} \right|$.

By which dividing the Fluent juft now found, and

and the Quotient $\frac{\frac{2a}{3b} \times \overline{bb+yy}^{\frac{3}{2}} - 2ab^3}{\frac{ay}{b} \sqrt{bb+yy} + ab} \bigg| \frac{y + \sqrt{bb+yy}}{b}$

$$= \frac{2 \times \overline{bb+yy}^{\frac{3}{2}} - 2b^3}{3y\sqrt{bb+yy} + 3bb} \bigg| \frac{y + \sqrt{bb+yy}}{b} \text{ will be } =$$

Distance of the Centre of Gravity Z from the Centre C .

Now to construct the Expression: Make $AC(a) : PM\left(\frac{a}{b} \sqrt{bb+yy} = x\right) :: BC(b) : CR =$

$\sqrt{bb+yy}$. And again, $PM\left(\frac{a}{b} \sqrt{bb+yy}\right) : AC(a) :: BC(b) : CS = \frac{b^2}{\sqrt{bb+yy}}$. And moreover,

$PM\left(\frac{a}{b} \sqrt{bb+yy}\right) : AC(a) :: CS\left(\frac{b^2}{\sqrt{bb+yy}}\right) : CT = \frac{b^3}{bb+yy}$. And let CR, CT , tend both

the same way, viz. from C towards PM , but CS the contrary way. Then take CN equal to the Measure of the Ratio between $CB(b)$ and $ER(\sqrt{bb+yy}-y)$ to the Module CS

$\left(\frac{b^2}{\sqrt{bb+yy}}\right)$. Which will be thus expressed,

$\frac{bb}{\sqrt{bb+yy}} \bigg| \frac{b}{\sqrt{bb+yy}-y}$. This being done, make

$3PN\left(3y + \frac{3b^2}{\sqrt{bb+yy}} \bigg| \frac{b}{\sqrt{bb+yy}}\right) : 2TR$
 $\left(2\sqrt{bb+yy} - \frac{2b^3}{bb+yy}\right) :: CR(\sqrt{bb+yy}) : to$

a fourth

a fourth Proportional, which will be =

$$\frac{2 \times \sqrt{bb+yy}^{\frac{3}{2}} - 2b^3}{3y\sqrt{bb+yy} + 3bb} \left| \frac{y + \sqrt{bb+yy}}{b} \right| = CZ \text{ the Di-}$$

stance sought.

Note, The Construction of these two Examples are the same as Mr. Cotes has given in *Harmonia Mensurarum*, p. 25, 26. Part I.

EXAMPLE VIII.

86. **T**O find the Centre of Gravity of a right Cone and Pyramid.

The Centre of Gravity will be somewhere FIG. 43. in the Axis *AB*.

Now if *AP* be = *x*, and *BC* = *a*, and *AB* = *b*, we shall have *PM* = $\frac{ax}{b}$. And the Ra-

tio of the Radius to the Periphery being that of *r* to *p*, the Fluxion of the Weight will be

$\frac{paaxx}{2rbb}x$, and the Momentum of it $\frac{paax^3}{2rbb}x$. Con-

sequently the Sum of the Momentums $\frac{paax^4}{8rbb}$,

divided by the Sum of the Weights $\frac{paax^3}{6rbb}$,

will give us $\frac{3}{4}x = AG$, the Distance of the

Centre of Gravity of the Part *AMPm* of the

Cone; and so $\frac{3}{4}AB$ is the Distance of the

Centre of Gravity of the whole Cone from

the Vertex *A*.

O o z

Much

Much after the same way you will find the Distance of the Centre of Gravity of a Pyramid from the Vertex to be $\frac{1}{4}$ of the Axis from the Vertex.

EXAMPLE IX.

FIG. 44. 87. **T**O find the Centre of Gravity of any Segment of a Sphere.

Let $AC=r$, $AP=x$; then the Fluxion of the Weights will be $px\dot{x} - \frac{px^2\dot{x}}{2r}$, and that of the Momentums $px^2\dot{x} - \frac{px^3\dot{x}}{2r}$, and the Sum of the Momentums $\frac{px^2}{3} - \frac{px^4}{8r}$; which being divided by $\frac{pxx}{2} - \frac{px^3}{6r}$, the Sum of the Weights, and the Quotient $\frac{8rx-3xx}{12r-4x}$ will be the Distance of the Centre of Gravity from A , of the Segment of a Sphere generated by the Semi-segment AMP of a Semicircle about AP .

COROL.

88. **H**ENCE the Centre of Gravity of $\frac{1}{2}$ the Sphere will be $\frac{3}{8}r$: for here x becomes $=r$.

EXAMPLE X.

89. **T**O find the Centre of Gravity of a parabolick Conoid formed by the Revolution of a Parabola about its Axis.

Here $\frac{pxx}{2}$ is the Fluxion of the Weights, and $\frac{px^2\dot{x}}{2}$ the Fluxion of the Momentums, r being

ing the *Semi-latus Rectum*. The Sum of the Weights is $\frac{px^2}{4}$, and the Sum of the *Momentums* $\frac{px^3}{6}$; therefore this divided by $\frac{px^2}{4}$, and the Quotient will be $\frac{2}{3}x$. Which is the Distance of the Centre of Gravity from the Vertex *A* of the common Parabola.

EXAMPLE XI.

90. **T**O find the Centre of Gravity of a Solid formed by the Revolution of the parabolic Space *AMBD* about the Line *BT* parallel to the Axis.

Let *DB* or *AT* be $=r$, *AP* $=x$, *PM* $=y$; FIG. 65:
then will $\frac{pr}{2} - py + \frac{pyy}{2r}$ be $=$ Circle descri-

bed by *MQ*; and consequently $px^{\frac{1}{2}}\dot{x} - \frac{px^2\dot{x}}{2r}$ will be the Fluxion of the Weights, and $px^{\frac{3}{2}}\dot{x} - \frac{px^2\dot{x}}{2r}$ that of the *Momentums*. The

Fluent of the former Expression, or Sum of the Weights, is $\frac{2}{3}px^{\frac{3}{2}} - \frac{px^2}{4r}$, and the Sum of the *Momentums* $\frac{2}{5}px^{\frac{5}{2}} - \frac{px^3}{6r}$. Which divided by the Sum of the Weights, and the Quoti-

ent will be $\frac{24rx^{\frac{3}{2}} - 10xx}{40rx^{\frac{1}{2}} - 15x} = \frac{24rx - 10yx}{40r - 15y}$, be-

ing the Distance of the Centre of Gravity of the Part of the Solid generated by *APM*;
and

and when x becomes $= a$, and so $y = r$, the said Distance will be $= \frac{14}{25} a$.

EXAMPLE XII.

91. *TO find the Centre of Gravity of an hyperbolick Conoid ABG generated by the Rotation of the hyperbolick Space ABG about the transverse Axis aCAP.*

FIG. 41. Now calling $AC, 2b, AB, a$, and the Ordinate BG, r , the Fluxion of the Weights will be $\frac{prxxx + 2bprxx}{2aa + 4ab}$, and the Fluxion of the Momentums. A being the Centre of Motion, will be $\frac{prx^3x + 2bprxxx}{2aa + 4ab}$; the Fluent of which is $\frac{prx^4}{8aa + 16ab} + \frac{bprx^3}{3aa + 6ab} = \frac{3prx^4 + 8bprx^3}{24aa + 48ab}$. Which being divided by $\frac{prx^3 + 3bprxx}{6aa + 12ab} = \frac{4prx^3 + 12bprxx}{24aa + 48ab}$ the Sum of the Weights, and the Quotient $\frac{3xx + 8bx}{4x + 12b}$ will be the Distance of the Centre of Gravity from the Point A in the Axis AP of the Conoid form'd by the Revolution of the Part AMP of the Hyperbola; and when $x = a$, $\frac{3aa + 8ab}{4a + 12b}$ will be the Distance of the Centre of Gravity of the whole Solid from the Vertex A . Whence $3a + 8b : 4a + 12b :: a : \frac{3aa + 8ab}{4a + 12b}$.

EXAM-

EXAMPLE XIII.

92. *TO find the Centre of Gravity of an hyperbolick Conoid form'd by the Revolution of the hyperbolick Space AMBDC about CD the half of the conjugate Axis.*

The same being supposed as in *Art. 61.* the *FIG. 48.*

Fluxion of the Weights is $\frac{aapx^2 \dot{x} + aabbp\dot{x}}{2bbr}$,

and that of the *Momentums* with regard to *AC*, is $\frac{aapx^3 \dot{x} + aabbp\dot{x}x}{2bbr}$; the Fluents of which

is $\frac{aapx^4}{8bbr} + \frac{aabbp\dot{x}x}{4bbr} = \frac{px^4yy + 2bbpx^2y^2}{8rx^2 + 8bbr}$, by put-

ting $\frac{bbyy}{xx+bb}$ for *aa*. Now this divided by the Sum of the Weights, and the Quotient $\frac{3x^3 + 6bbx}{4xx + 12bb}$, will be the Distance of the Cen-

tre of Gravity of the Conoid generated by the Space *ACPM* from the Line or Axis *AC*;

and when *x* becomes = *b*, we shall have $\frac{9}{16}b$

for the Distance of the said Centre.

Therefore the Centre of Gravity of the whole Conoid is so situate in the Axis *AC*, that the Part from the Centre *C* to the Centre of Gravity is to *AC* as 9 to 16.

EXAMPLE XIV.

93. *TO find the Centre of Gravity of a Semi-spheroid generated by the Rotation of $\frac{1}{2}$ the Elliptick Space AMca about the Axis Aa.*

FIG. 40. Make $AC = a$, $Cc = r$, $AP = x$, $PM = y$. Now when PC is x , the Fluxion of the Weights will be $\frac{pyy\dot{x}}{2r}$; but from the Nature of the Ellipsis, (AP being $= x$) $yy : 2ax - xx :: rr : aa$. Whence $yy = \frac{2arrx - rrx^2}{aa}$; and so substituting this Value in $\frac{pyy\dot{x}}{2r}$, there arises $\frac{2aprx\dot{x} - prx^2\dot{x}}{2aa}$; and multiplying by x ; the Fluxion of the Momentums will be $\frac{2aprx^2\dot{x} - prx^3\dot{x}}{2aa}$; the Fluent of which, viz. $\frac{prx^3}{3a} - \frac{prx^4}{8aa} = \frac{8aprx^3 - 3prx^4}{24aa}$, divided by the Sum of the Weights $\frac{12aprx^2 - 4prx^3}{24aa}$, and the Quotient $\frac{8ax - 3xx}{12a - 4x}$ will be the Distance of the Centre of Gravity of the partial Solid generated by the Space APM from the Point A ; and when x becomes $= a$, the said Distance will be $\frac{5}{8}a$; that is, the Centre of Gravity of the Semi-spheroid will be distant from A by $\frac{5}{8}a$.



SECTION VII.

Of the Use of Fluxions in finding the Centres of Percussion of Figures.

DEFINITION.

94. **T**HE Centre of *Percussion*, or *Oscillation* of a Figure in Motion, is that Point in which all the Forces of the same are consider'd as united together in one; so that if the said Figure meets any Obstacle contrary to the Motion thereof, it strikes the Obstacle with a greater Force than any other Point of the Figure.

In order to this, it is necessary that the Parts of the Figure do constantly alter their Disposition to move; that they separate their Quantity of Motion, not as in the Centre of Gravity in the Ratio of the Spaces run thro'; but in a Ratio compounded of their Velocities, and the Distances of that Centre reciprocally proportional to the said Velocities; or, which is the same thing, into equal Quantities of Motion on each Side that Point. Therefore the Centre of Percussion is the same with regard to Velocities, as the Centre of Gravity is with respect to Weights: and as in finding the Centre of Gravity, we divide the Sum of the *Momentums* by the Sum of the Weight; so to find the Centre of Percussion, we must multi-

P p

ply

ply the Sum of the *Momentums* by straight Lines equal or proportional to the Spaces moved thro', and divide the Product by the Sum of the *Momentums*. Whence the

General Rule for finding the Centre of Percussion of a Figure that revolves about a given Point or Axis, is to multiply all the small Parts of which the Figure consists, (that is, the Area or Solidity of it) looked upon as so many Weights, by the Squares of their Distances from the Point of Suspension, and divide the Product by the Product of the same Weights into the Distances from the Axis of Motion, and the Quotient will be the Distance of the Centre of Percussion from the Point or Axis of Motion.

FIG. 59. Hence if AP be $= x$, $MN = 2y$, $Pp = \dot{x}$, the *Momentum* of the whole small Weight $MNnm$ will be $= 2y x \dot{x}$. Consequently the Distance of the Centre of Percussion from the Point A is $=$ to the Fluent of $2y x^2 \dot{x}$ divided by the Fluent of $2y x \dot{x}$. Consequently, if from the Equation of the Figure you get the Value of y , and put it in those Fluxions, and then find the Fluents of them, you will get the Distance of the Centre of Percussion from the Point A . This will be evident from the following Examples.

EXAMPLE I.

FIG 65. 95. **T**O find the Centre of Percussion of a right Line AB , moving about one End A thereof.

Now if the said Line be conceived to be divided into an infinite Number of equal small Parts

Pp

$Pp(\dot{x})$ (AB being a , and AP, x), it is manifest that in equal times they will describe equal Arches of concentrick Circles, that will be to each other as the Distances from the Point A : But the Velocities wherewith the said Arches are moved through, are proportional to the said Arches; and so the Velocities are as the said Distances.

Now $x\dot{x}$ will be the Fluxion of the *Momentums*, which multiplied by x representing the Velocities, and there will arise $x^2\dot{x}$ for the Fluxion of Forces; the Fluent of which $\frac{x^3}{3}$ being divided by the Sum of the *Momentums* $\frac{x^2}{2}$, and the Quotient $\frac{2}{3}x$ will be the Distance of the Centre of Percussion of the Part AP of the Line from the Point A ; and the Centre of Percussion of the whole Line will be $\frac{2}{3}a$, by making $x = a$.

E X A M P L E II.

96. **T**O find the Centre of Percussion of a Rect-angle $RISH$ moving about one of the Sides RI .

If $RI = SH$ be $= a$, $AP = x$; then will **FIG. 66.**
 $Pp = \dot{x}$ be the Fluxion of the Area, and one of the small Weights will be $= a\dot{x}$, and the *Momentum* of it will be $ax\dot{x}$. Whence the Fluent of $ax^2\dot{x}$ divided by the Fluent of $ax\dot{x}$ (or $\frac{\frac{1}{2}ax^3}{\frac{1}{2}ax^2}$), and the Quotient $\frac{2}{3}x$ will be the Distance of the Centre of Percussion of the
 P p 2 Part

Part $RCDI$ of the Parallelogram from the Side RI ; and if for x be put the Altitude $RS = b$ of the whole Rectangle, the Distance of the Centre of Percussion from the Side RI will be $= \frac{2}{3}b$.

EXAMPLE III.

97. **T**O find the Centre of Percussion of an Isosceles Triangle SAH moving about the Line RI passing thro' the Vertex A , and parallel to the Base SH .

FIG. 66. Let the Altitude AE be $= a$, $AP = x$, $EH = \frac{1}{2}b$, $PL = y$. Then $AP(x) : PL(y) :: AE(a) : EH = \frac{1}{2}b$. Whence $ay = \frac{1}{2}bx$, and $y = \frac{bx}{2a}$. Now the Fluent of $y x^2 \dot{x} =$ Fluent of $\frac{bx^3 \dot{x}}{2a}$ is $= \frac{bx^4}{8a}$; and the Fluent of $y x \dot{x} =$ Fluent of $\frac{bx^2 \dot{x}}{2a}$ is $= \frac{bx^3}{6a}$. Whence the Fluent of $y x^2 \dot{x}$, divided by that of $y x \dot{x}$, or $\frac{6abx^4}{8abx^3}$ is $= \frac{6}{8}x = \frac{3}{4}x$.

Now if for x you substitute the whole Altitude $AE = a$, the Distance of the Centre of Percussion of the whole Triangle ASH from the Vertex A will become $\frac{3}{4}a = \frac{3}{4}AE$.

EXAM.

EXAMPLE IV.

98. **TO** find the Centre of Percussion of an Isosceles Triangle *SAH* moving about the Base *SH*.

Let all things be as in the last Example, *FIG. 66.* then will *PE* be $= a - x$. Whence the Flu-

ent of $y \dot{x}^2 \dot{x} =$ Fluent of $\frac{b \dot{x} \dot{x}}{2a} \times a - x^2 (=$ Flu-

ent of $\frac{1}{2} a b \dot{x} \dot{x} = b \dot{x}^2 \dot{x} + \frac{b \dot{x}^3 \dot{x}}{2a})$ is $= \frac{1}{4} a b \dot{x}^2 -$

$\frac{1}{3} b \dot{x}^3 + \frac{b \dot{x}^4}{8a}$; and the Fluent of $y \dot{x} \dot{x} =$ Fluent of

$\frac{b \dot{x} \dot{x}}{2a} \times a - x =$ (Fluent of $\frac{1}{2} b \dot{x} \dot{x} - \frac{b \dot{x}^2 \dot{x}}{2a}) =$

$\frac{1}{4} b \dot{x}^2 - \frac{b \dot{x}^3}{6a}$. Whence the Quotient of the

Fluent of $y \dot{x}^2 \dot{x}$ divided by that of $y \dot{x} \dot{x}$

$$\left(\frac{\frac{1}{4} a b \dot{x}^2 - \frac{1}{3} b \dot{x}^3 + \frac{b \dot{x}^4}{8a}}{\frac{1}{4} b \dot{x}^2 - \frac{b \dot{x}^3}{6a}} \right)$$

$$= \frac{24a^2 b \dot{x}^2 - 32ab \dot{x}^3 + 12b \dot{x}^4}{16a}$$

$$\text{is} = \frac{6ab \dot{x}^2 - 4b \dot{x}^3}{24a}$$

$$= \frac{6a^2 b \dot{x}^2 - 16ab \dot{x}^3 + 6b \dot{x}^4}{12ab \dot{x}^2 - 8ab \dot{x}^3} = \frac{6a^2 - 8ax + 3x^2}{6a - 4x}$$

$=$ Distance of the Centre of Percussion of the Segment *SZVH* from the Base *SH*.

Now if for *x* you substitute *a*, we shall have the Distance of the Centre of Percussion of

of the whole Triangle $SAH =$

$$\left(\frac{6a^2 - 8a^2 + 3a^2}{6a - 4a} \right) = \frac{1}{2}a = \frac{1}{2}AE.$$

EXAMPLE V.

99. **T**O find the Centre of Percussion of a parabolick Space, moving about a Line passing thro' the Vertex parallel to the Base.

Now calling the Absciss x , and the whole Height a , and the Fluxion of the *Momentums* will be $x^{\frac{3}{2}}\dot{x}$, and that of the Forces $x^{\frac{5}{2}}\dot{x}$; the Fluent of which will be $\frac{2}{7}x^{\frac{7}{2}}$. Which divided by $\frac{2}{5}x^{\frac{5}{2}}$ the Sum of the *Momentums*; the

Quotient will be $\frac{5}{7}x =$ Distance of the Centre of Percussion of the Part of the Parabola whose Height is x from the Vertex; and when x becomes equal to a , the Distance of the Centre of Percussion of the whole Parabola will be $\frac{5}{7}a$.

SCHOLIUM.

100. **I**F the Centre of Percussion of a Parabola of any Kind be sought, you will have $\frac{2m+1}{3m+1}a =$ Distance of the Centre of Percussion from the Vertex; where m is the Exponent of the Power of the Ordinate of the Parabola. So that if m be $= 2$, the Distance will be $\frac{5}{7}a$, as in the common Parabola. If

m be

m be $= 3$, as in the cubick Parabola, the Distance will be $\frac{7}{10}a$, &c.

EXAMPLE VI.

101. **T**O find the Centre of Percussion of a parabolick Space moving about its Base DD . Fig. 38.

Multiply the Fluxion of the Weights $x^{\frac{1}{2}} \dot{x}$ by $a - x$, and we get $ax^{\frac{1}{2}} \dot{x} - x^{\frac{1}{2}+1} \dot{x} =$ the Fluxion of the *Momentums*; which being again multiplied by $a - x$, and the Fluxion of the Forces will be $aa x^{\frac{1}{2}} \dot{x} - 2ax^{\frac{1}{2}+1} \dot{x} + x^{\frac{1}{2}+2} \dot{x}$, the Fluent of which, viz. $\frac{2}{3}aa x^{\frac{1}{2}+1} - \frac{4}{5}ax^{\frac{1}{2}+2}$

$$+ \frac{2}{7}x^{\frac{1}{2}+3} = \frac{70aa x^{\frac{1}{2}+1} - 84ax^{\frac{1}{2}+2} + 30x^{\frac{1}{2}+3}}{105}$$

being divided by the Sum of the *Momentums* $= \frac{10ax^{\frac{1}{2}+1} - 6x^{\frac{1}{2}+2}}{15}$, and the Quotient will

be $\frac{35aa - 42ax + 15xx}{35a - 21x}$, the Distance of the

Centre of Percussion of the Space $DMND$ from the Base DD ; and putting x for a , the Distance of the Centre of Percussion of the whole Parabola from the Base DD will be

$$\frac{4}{7}a = \frac{4}{7}AB.$$

EXAMPLE VII.

102. **T**O find the Centre of Percussion of a Cylinder AB moving about the End A thereof. Fig. 67.

Now

Now it is evident that the Velocities of all the small equal Parts of this Solid will be to each other, on account of the equal times, in the same Ratio as the Spaces rm thro' that is, as the Arches described in their Motion, or as the Radii or Distances from the Point of Suspension. Now let the Axis AB of the Cylinder be $= a$, any Part $AP = x$, the Periphery of the Base $= p$, and the Radius of it $= r$; then the Fluxion of the *Momentums* will be $\frac{prxx}{2}$, and the Fluxion of the Forces $\frac{prx^2x}{2}$,

the Fluent of which is $\frac{prx^3}{6}$. Which being divided by the Sum of the *Momentums* $= \frac{prx^2}{4}$, and the Quotient $\frac{2}{3}x$ will be the Distance of the Centre of Percussion of the Part of the Cylinder, whose Altitude is AP from the Point A ; and $\frac{2}{3}a$ will be the Distance of the Centre of Percussion of the whole Cylinder from the said Point A .

EXAMPLE VIII.

103. **T**O find the Centre of Percussion of a Cylinder moving about the Point R in the Axis continued out.

FIG. 67. Make $RB = a$, $RA = b$, $AP = x$; then $RP = b + x$, and $AB = a - b$. Whence $\frac{bprx + prxx}{2}$ is the Fluxion of the *Momentums*:

But the Velocities of the little equal Weights of the Solid are to one another as the Arches described from the Point R by those

those Weights; so the Forces are to each other as the right Lines that fill up a Trapezium described by the Cylinder. Whence multiplying the Fluxion of the *Momentums* by $b+x$; and we get $\frac{bbprx + 2bprxx + prx^2x}{2} =$ Fluxion

of the Forces; the Fluent whereof is $= \frac{bbprx}{2} + \frac{bprx^2}{2} + \frac{prx^3}{6}$. Which divided by the

Sum of the *Momentums* $= \frac{bprx}{2} + \frac{prx^2}{4}$, and

the Quotient $\frac{6bb + 6bx + 2xx}{6b + 3x}$ will be the Di-

stance of the Centre of Percussion of the Part whose Altitude is AP from the Point R ; and

$\frac{2aa + 2ab + 2bb}{3a + 3b}$ will be the Distance of the

Centre of Percussion of the whole Cylinder from the Point R , since then x becomes $= a-b$.

Hence it is easy to find on what Part of a Cylindrical Stick a Man ought to strike in order to gain the greatest Blow possible, supposing AR represents the Man's Arm, and AB the Stick.

EXAMPLE IX.

104. *TO find the Centre of Percussion of a Cone moving about its Vertex A.*

The Fluxion of the *Momentums* will be FIG. 43; $\frac{prx^3x}{2aa}$, and the Fluxion of the Forces $\frac{prx^4x}{2aa}$;

the Fluent $\frac{prx^5}{10a^2}$ of which being divided by the Sum of the *Momentums* $= \frac{prx^4}{8aa}$, and the Quotient $\frac{4}{5}x$ will be the

Qq

Distance

Distance of the Centre of Percussion of the Part of the Cone, whose Altitude is AP from the Vertex A , and $\frac{4}{5}a$ will be the Distance of the Centre of Percussion of the whole Cone from the Vertex A .

Note, This Centre of Percussion is the same as the Centre of Gravity of the Complement of a cubick Parabola; because the Forces are as right Lines which fill up that Complement.

EXAMPLE X.

105. *TO find the Centre of Percussion of a Sphere moving about a Point A in the End of a Diameter AD.*

FIG. 44. Let the Radius be $=r$, the Periphery $=p$, and $AP=x$; then the Fluxion of the *Momentums* will be $px^2\dot{x} - \frac{px^3\dot{x}}{2r}$, and the Fluxion of the Forces $px^3\dot{x} - \frac{px^4\dot{x}}{2r}$; and the Fluent of this will be $=\frac{px^4}{4} - \frac{px^5}{10r}$; which being divided by the Sum of the *Momentums* $=\frac{px^3}{3} - \frac{px^4}{8r}$, and the Quotient $\frac{30rx - 12xx}{40r - 15x}$ will be the Distance of the Centre of Percussion of the Segment of the Sphere, whose Height is x , from the Point A ; and $\frac{6}{5}r$ will be the Distance of the Centre of Percussion of the whole Sphere from that Point.

EXAMPLE

EXAMPLE XI.

106. *TO find the Centre of Percussion of a Parabolick Conoid moving about the Vertex.*

The Fluxion of the *Momentums* will be $\frac{px^2 \dot{x}}{2r}$, and the Fluxion of the Forces $\frac{px^3 \dot{x}}{2r}$. The

Fluent of which is $= \frac{px^4}{8r}$; which being divi-

ded by $\frac{px^3}{6r}$, and $\frac{3}{4}x$ will be the Distance of the Centre of Percussion of the Part of the Conoid, whose Altitude is x from the Vertex, and $\frac{3}{4}a$ will be the Distance of the Centre of Percussion of the whole Conoid from the Vertex.

EXAMPLE XII.

107. *TO find the Centre of Percussion of a Spheroid moving about one End of the transverse Axis.*

The same things being supposed as in Art. 93. the Fluxion of the *Momentums* will be $\frac{2aprx^2 \dot{x} - prx^3 \dot{x}}{2aa}$, and the Fluxion of the For-

ces $\frac{2aprx^3 \dot{x} - prx^4 \dot{x}}{2aa}$; the Fluent of which, viz.

$\frac{prx^4}{4a} - \frac{prx^3}{10aa}$, being divided by the Sum of the

Momentums $= \frac{prx^3}{3a} - \frac{prx^4}{8aa}$, and the Quotient

Qq 2

$\frac{30ax - 12xx}{40a + 15x}$ will be the Distance of the Centre of Percussion of the Part of the Spheroid, whose Altitude is x from the Point of Motion, and $\frac{5}{6}a$ will be the Distance requir'd.





S E C T. VIII.

Of the Resolution of some miscellaneous Problems by Fluxions.

P R O B. I.

108. **T**O find a Line, wherein the Subtangent is equal to the Semi-ordinate.

In all Cases it is plain that $\frac{y\dot{x}}{y}$ is an Expression of the Subtangent; and so from the Condition of the Problem $\frac{y\dot{x}}{y} = y$, and $y\dot{x} = yy$; that is, $\dot{x} = y$, and the Fluent of each Side will be $x = y^2$. Whence the Line sought is the Hypothenufe of a right-angled Equicrural Triangle; a Line bisecting the right Angle being looked upon as the Axis. But if x be the Arch of a Circle, then will the Line sought be a Cycloid.

P R O B. II.

109. **T**O find a Curve, whose Subtangent is equal to twice the Square of the Semi-ordinate divided by a constant Quantity: suppose a .

The

The Subtangent is $= \frac{y\dot{x}}{\dot{y}}$. Whence $\frac{y\dot{x}}{\dot{y}}$ must be $= \frac{2yy}{a}$. Consequently $ay\dot{x} = 2y^2\dot{y}$, that is, $a\dot{x} = 2y\dot{y}$; and finding the Fluents of each Side, we have $ax = yy$; therefore the Curve sought is the *Apollonian Parabola*.

PROB. III.

110. *TO find a Curve, whose Subtangent is a third Proportional to some standing Quantity a lessen'd by the Absciss, and the Semi-ordinate.*

Here $a - x : y :: y : \frac{y\dot{x}}{\dot{y}}$; and multiplying the Means and Extremes we get $\frac{ay\dot{x} - yx\dot{x}}{\dot{y}} = yy$, and $ay\dot{x} - yx\dot{x} = y^2\dot{y}$; that is, $a\dot{x} - x\dot{x} = y\dot{y}$; and finding the Fluents of each Side, we have $ax - \frac{1}{2}xx = \frac{1}{2}yy$, or $2ax - xx = yy$; therefore the Curve sought is a Circle whose Radius is $= a$.

PROB. IV.

111. *TO find a Curve, whose Subtangent is an invariable Line.*

Here $\frac{y\dot{x}}{\dot{y}}$ must be $= a$, and so $\dot{x} = \frac{a\dot{y}}{y} = ay^{-1}\dot{y}$. Whence the Fluent of $\dot{x}$, viz. x must be $=$ Fluent of $ay^{-1}\dot{y}$; and multiplying $ay^{-1}\dot{y}$ by a , we shall have $a^2y^{-1}\dot{y}$, which is the Fluxion of an Hyperbola between the Asymptotes. And so if y be taken for an Absciss, the correspondent Semi-ordinate $x =$ Fluent of $ay^{-1}\dot{y}$ will be equal to the Asymptotical Hyperbolical Space

divided by the invariable Quantity a , being the Side of the Power of the Hyperbola.

PROB. V.

112. *A* Number being given: to find the Logarithm of it.

Let the Ordinate of the Logarithmical FIG. 68.
Curve AB be $= 1$ = Subtangent; then PM will represent a Number greater than 1, and QN a Number less than 1; AP the Logarithm of the Number greater than 1, and AQ the Logarithm of the Number less than 1.

Now let the Difference between AB and PM be $= y$; then will PM be $= 1 + y$. Whence AP , the Logarithm of a Number

greater than 1, will be the Fluent of $\frac{y}{1+y} = y - yy + y^2y - y^3y + y^4y, \&c.$ Which Fluent is $= y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4, \&c.$ = Logarithm of a Number greater than 1.

Again, if the Difference between AB and QN be y ; then will $QN = 1 - y$. And so AQ , or the Logarithm of a Number less than 1, will be = Fluent of $\frac{-y}{1-y} = -y - yy - y^2y - y^3y - y^4y, \&c.$ the Fluent of which will be $= -y - \frac{1}{2}y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4, \&c.$ being the Logarithm of a Number less than 1.

SCHOLIUM.

113. **I**F the Side AB or BC of the Power of FIG. 69.
an Hyperbola be $= 1$, and $BP = y$; then will $AP = 1 + y$, and the Asymptotical Hyperbolical Space will be $= y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4$

$-\frac{1}{4}y^4$, &c. And if BQ be $= y$, then will AQ be $= 1 - y$, and the Fluxion $\frac{-y\dot{y}}{1-y}$ will be that of the Hyperbolick Asymptotical Space. Whence the Space will be $= -y - \frac{1}{2}y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4$, &c.

Whence Logarithms may be represented also by the Hyperbola: for if the Side AB of the Power of the Hyperbola be $= 1$, the Absciss AP is a Number greater than 1, and the Asymptotical Space $BCMP$ is the Logarithm of a Number greater than 1; in like manner the Absciss AQ is a Number less than 1, and the Asymptotical Hyperbolical Space $QNCB$ is the Logarithm of a Number less than 1. Again, if $y = 1$, then will $1 + y = 2$; and so the Hyperbolical Logarithm of 2 will be $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8}$, &c. But these Series converge very slowly; which may be remedied by substituting $\frac{x}{1+x}$ for y .

Note, These Hyperbolical Logarithms are the same as *Napier's*; and so are different from *Briggs's*, which we commonly used; but they may be reduced to *Briggs's*: being to his, as the Hyperbolical Logarithm of 10, viz. 2.302585092994, &c. to *Briggs's* Logarithm of 10, that is, 1.000000000000, &c.

SCHOLIUM II.

114. IF z be an odd Number, whose Logarithm is sought; the Numbers $z - 1$ and $z + 1$ will be even; and so their Logarithms and the Difference of their Logarithms will be given, which let be y . Likewise the Logarithm of a Number being a geometrical

I

Mean

Mean between the Numbers $z-1$ and $z+1$, viz. half the Sum of the Logarithms. The

Series $y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{15120z^7} +$

$\frac{13}{25200z^9}$, &c. will be the Logarithm of the

Ratio, which the geometrical Mean between the Numbers $z-1$ and $z+1$ has to an arithmetical Mean, viz. the Number z .

PROB. VI.

115. **I**F a Body freely descends by its own Gravity from the Point A along two inclined Plains AB, AC, to the Points B and C; it is required to find the Proportion of the Times of Description. FIG. 70.

Let ADE be a vertical Line, and BD, CE horizontal ones. Call AD, a , AE, b , DB, x , and EC, z . Now it is evident, that the Time of a Body's describing an infinitely small Part of any Line, as AB or AC may be taken for the Fluxion of the Time of its describing the whole Lines AB and AC. This being premised, $AB = \sqrt{aa + xx}$, and $AC = \sqrt{bb + zz}$.

The Fluxion of the former will be $\frac{x \dot{x}}{\sqrt{aa + xx}}$,

and that of the latter $\frac{z \dot{z}}{\sqrt{bb + zz}}$

Again, the Velocity of the Body describing the infinitely small Part of AB expressed by

$\frac{x \dot{x}}{\sqrt{aa + xx}} = Bb$, and that of AC expressed by

$\frac{z \dot{z}}{\sqrt{bb + zz}} = Cc$, will be equal to the respective

R r

Velo-

Velocities that the Body falling perpendicularly from A will have in the Points D and E ; the Velocities of the Description of the very small Parts Bb , Cc being looked upon as equal: which are proportional to $\sqrt{AD}$ ($\sqrt{a}$), and $\sqrt{AE}$ ($\sqrt{b}$). Therefore the Times of de-

scribing Bb and Cc are as $\frac{x\dot{x}}{\sqrt{a^3 + ax}}$ and

$\frac{z\dot{z}}{\sqrt{b^3 + bz^2}}$ (the Times of the Descriptions of

any Spaces being as the Spaces directly, and the Velocities inversely.) Consequently the Time of describing AB to the Time of describing AC , will be as the Fluent of the former Fluxion to that of the latter. But these Fluents are easily had from the fifth Form of Mr. Cotes's Tables, by making $\theta=1$, $n=2$, &c.

that of the former Fluxion being $\sqrt{\frac{aa+xx}{a}}$,

and that of the other $\sqrt{\frac{bb+zz}{b}}$. So that the

Time of the Description of AB to that of the

Description of AC will be as $\frac{AB}{\sqrt{AD}} \left(\sqrt{\frac{aa+xx}{a}} \right)$

to $\frac{AC}{\sqrt{CE}} \left(\sqrt{\frac{bb+zz}{b}} \right)$.

PROB. VII.

116. **T**O find the Nature of the Curve BC being such, that a Body freely falling by its own Gravity perpendicularly from the Point A to B , and thence continuing to move on along the said Curve, shall descend equal Spaces in equal Times.

FIG. 71.

Let BD be the Axis; then let $AB=a$, the Absciss $BP=x$, and the Ordinate $PM=y$.

Now

Now the Fluxion of the Ordinate $\dot{x}$ is $=0$ in the Point B the Beginning of the Curve, because it is convex next to the Axis, and the Axis is a Tangent to it in B ; therefore Mm ($\sqrt{\dot{x}^2 + \dot{y}^2}$) is $=\dot{x}$ at the Point B ; therefore the Time of the Bodies describing the Arch Bb is as $\frac{\dot{x}}{\sqrt{a}}$; since the Times are as the Spaces directly, and Velocities inversly: and the Velocities acquir'd in B and M being those acquir'd by the Fall from A to the Points B and M , (which Velocities are in the subduplicate Ratio of AB to AP), and the Time of the Description of the small Arch Mm is as $\sqrt{\frac{\dot{x}^2 + \dot{y}^2}{a+x}}$. But since from the Condition of the Problem, during the Description of the Curve, the Body falls equal Spaces in equal Times; therefore $\frac{\dot{x}}{\sqrt{a}}$ must be $=\sqrt{\frac{\dot{x}^2 + \dot{y}^2}{a+x}}$; and squaring both Sides, there arises $\frac{\dot{x}^2 + \dot{y}^2}{a+x} = \frac{\dot{x}^2}{a}$. Whence $a\dot{x}^2 + a\dot{y}^2 = a\dot{x}^2 + x\dot{x}^2$; that is, $a\dot{y}^2 = x\dot{x}^2$; and extracting the Root of both Sides, we get $\sqrt{ax}\dot{y} = \sqrt{x}\dot{x}$. Lastly, finding the Fluents, and $ay^2 = \frac{4}{9}x^3$, or $\frac{2}{3}ay^2 = x^3$. Which is the Equation of the Curve; and so it is a Semicubical Parabola.

Otherwise:

Let $AP = x$, $PM = y$, the Velocity at the End of the Fall from A to $M = v$, and $z =$ Time of the Fall to M . Now from the Principles of Mechanicks $\frac{Mm}{v}$ is as $\dot{z}$, the Time
R r 2 of

of Description of the small Arch Mm , that is, $\frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{v}$ is as $\dot{z}$. Whence if a be a proper

standing Quantity, we shall have $a\sqrt{\dot{x}^2 + \dot{y}^2} = v\dot{z}$. But v is as $\sqrt{x}$, or as $\sqrt{ax}$; and so we may take $v = \sqrt{ax}$. Also z is as $x - a$; since from the Condition of the Problem, the Time is as the Altitudes from whence the Body falls. Whence $x - a$ may be taken for z , and $\dot{x}$ for $\dot{z}$; so that substituting $\dot{x}$ for $\dot{z}$, and $\sqrt{ax}$ for v in the Equation $a\sqrt{\dot{x}^2 + \dot{y}^2} = v\dot{z}$, and there comes out $a\sqrt{\dot{x}^2 + \dot{y}^2} = \dot{x}\sqrt{ax}$. Whence $a^2\dot{x}^2 + a^2\dot{y}^2 = ax\dot{x}^2$, and $a^2\dot{y} = \dot{x}\sqrt{x - a}$. Then finding the Fluent of each Side, (which may be easily done from the little Table of Curves that may be squared page 34. or from the third Table of Forms of Mr. Cotes) and a^2y will be $= \frac{2ax - 2a^2}{3} \sqrt{ax - a^2}$, or $ay = \frac{2x - 2a}{3} \sqrt{ax - aa}$.

Now making $n = x - a$, and we have $\frac{2n}{3} \sqrt{an}$,

or $a^2y^2 = \frac{4an^3}{9}$, or $\frac{9}{4}ay^2 = n^3$: therefore the Curve sought is a second cubical Parabola, AB being $= a$, and $BP = n$.

P R O B. VIII.

FIG. 72. 117. *TO find the Law of Refraction, admitting this Principle, viz. that Nature in all its Operations takes the shortest ways.*

Because Light cannot move in different Mediums with the same Velocity; let the Ratio of the Velocity of the Light during its Motion from A to B , where it begins to be refracted,

fracted, to its Velocity, while it is refracted in its Motion from B to C , be expressed by $\frac{m}{n}$;

then the Times of the Descriptions of the Lines AB , BC will be as $n \times AB$ to $m \times BC$.

Let fall the Perpendiculars AQ , CP , and make $AQ = a$, $CP = b$, $PQ = c$, $PB = x$; then will $BQ = c - x$, and consequently BC

$= \sqrt{bb + xx}$, and $AB = \sqrt{aa + cc - 2cx + xx}$.

Whence the Time in which $AB + BC$ is moved thro', is $= m\sqrt{bb + xx} + n\sqrt{aa + cc - 2cx + xx}$, which must be a *Minimum*; and so the Fluxion

thereof, viz. $\frac{mxx}{\sqrt{bb + xx}} + \frac{nx}{\sqrt{aa + cc - 2cx + xx}}$

$= 0$. Whence $\frac{mx}{\sqrt{bb + xx}} = \frac{nx}{\sqrt{aa + cc - 2cx + xx}}$,

that is, $\frac{m \times PB}{BC} = \frac{n \times BQ}{AB}$. Make $BC = AB$,

then will $m \times PB = n \times BQ$, and consequently $m:n :: BQ:PB$.

Whence if BA or BC be taken for the Radius, BQ will be the Sine of the Angle A , and PB the Sine of the Angle C ; that is, since AQ and PC are parallel to DE , PB is the Sine of the Angle CBE , and BQ the Sine of ABD , viz. PB is the Sine of the refracted Angle, BQ the Sine of the Angle of Incidence. Whence the Sine of the Angle of Incidence is to the Sine of the refracted Angle in a constant Ratio, viz. that of the Velocity of Light before Refraction to the Velocity during its Refraction.

PROB.

PROB. IX.

FIG. 73. 118. **T**O find the Angle BCD, in which a Body from A obliquely striking a Plain in C may be reflected to a given Point B, so as to pass from the given Point A to the given Point C the shortest way.

From A, B let fall the Perpendiculars AE, BD. Let $AE = a$, $BD = b$, $ED = c$, and $EC = x$; then $CD = c - x$, and $AC = \sqrt{aa + xx}$; also $CB = \sqrt{bb + cc - 2cx + xx}$; and so $AC + CB$ must be *Minimum*, that is, $\sqrt{aa + xx} + \sqrt{bb + cc - 2cx + xx}$. Whence the Fluxion of it must be made equal to 0; therefore $\frac{\dot{x}}{\sqrt{aa + xx}} + \frac{xx - c\dot{x}}{\sqrt{bb + cc - 2cx + xx}} = 0$; and so $x\sqrt{bb + cc - 2cx + xx} + x - c \times \sqrt{aa + xx} = 0$. Whence $x\sqrt{bb + cc - 2cx + xx} = c - x \times \sqrt{aa + xx}$; that is, $EC \times CB = CD \times AC$; and so $EC : AC :: CD : CB$. Consequently (by Prop. 7. lib. 6. Eucl.) the Triangles AEC, BDC are equiangular: Whence the Angle ACE must be = Angle BCD.

PROB. X.

FIG. 74. 119. **I**F a thin Fluid consists of equal Particles freely disposed at equal Distances from each other; it is required to find that Frustum of a Cone, which of all others of the same Base Aa, and Altitude BC, moving in that Fluid according to the Direction of the Axis, with the lesser Base Dd foremost, that shall have the least Resistance.

It

It is the same thing to consider the *Frustum* at rest, and the Particles to move against it with the same Velocity.

Draw DE parallel to BC . Let the given Altitude $BC = b$, the Radius of the given Base $= a$, and $AE = x$. Now it is well known that the Effect of any Particle of the Fluid striking the Surface AD of the *Frustum* obliquely in the Direction DE to move it according to the same Direction, is to the Effect of the same Particle striking directly against the Annulus generated by the Line AE (while AB revolves about) to move it in the Direction DE , as the Square of the Sine of the Angle of Incidence FEV or ADE to the Square of the Radius. And since here the Angle of Incidence is invariable, the Effect of all the Particles striking the Superficies of the *Frustum* generated by AD , will be to the Effect of all the Particles that can strike the *Annulus* aforesaid is in that Proportion, that is, the Resistance of the Superficies of the *Frustum* generated by AD , is to the Resistance of the *Annulus* generated by AE , as the Square of the Sine of the Angle of Incidence is to the Square of the Radius.

Whence if BC (b) be made the Radius, the Sine of the Angle of Incidence will be

$$\frac{bx}{\sqrt{b^2 + x^2}}, \text{ for } AD (\sqrt{b^2 + x^2}) : AE (x) :: BC (b)$$

$$: \frac{bx}{\sqrt{b^2 + x^2}}.$$

Now if the aforesaid *Annulus* be made the Resistance of itself, then will the Circle described by BE be the Resistance of itself also; but since Circles are to each other as the Squares of their Radii, therefore the Resistance of the *Annulus* is to the Resistance

of the Circle, as $\overline{AB}^2 - \overline{BE}^2 (2ax - x^2)$ is to $\overline{BE}^2 (a^2 - 2ax + x^2)$. Which Quantities now let represent the respective Resistances of the *Annulus* and Circle.

Then $\overline{BC}^2 (b^2) : \frac{b^2 x^2}{b^2 + x^2}$ Square Sine :: $\overline{AB}^2 - \overline{BE}^2 (2ax - x^2) : \frac{2ax^3 - x^4}{b^2 + x^2}$ = Resistance of

the Superficies of the *Frustum* generated by AD . To which adding the Resistance of the lesser Base Dd (being $a^2 - 2ax + x^2$) and the Resistance of the whole *Frustum* will be

$$\frac{2ax^3 - x^4}{b^2 + x^2} + a^2 - 2ax + x^2 = \frac{a^2 b^2 - 2ab^2 x + b^2 x^2 + a^2 x^2}{b^2 + x^2}.$$

The Fluxion of which must be a *Minimum*; therefore $\frac{2ab^4 x \dot{x} + 2ab^2 x^3 \dot{x} - 2ab^4 \dot{x}}{(b^2 + x^2)^2} = 0$. Whence $x^2 +$

$$\frac{b^2}{a} x = b^2; \text{ and so } x = \frac{b}{2a} \sqrt{b^2 + x^2} - \frac{b^2}{2a}.$$

But because the Triangles AED , ABV are similar, therefore $AE(x) = \frac{b}{2a} \sqrt{b^2 + x^2} - \frac{b^2}{2a} : BC$

$$(b) :: AB(a) : BV = \frac{a^2}{\frac{1}{2} \sqrt{4a^2 + b^2} - \frac{1}{2} b}, \text{ which}$$

is $= \frac{1}{2} \sqrt{4a^2 + b^2} + \frac{1}{2} b^2$; since $\frac{1}{2} \sqrt{4a^2 + b^2} - \frac{1}{2} b \times \frac{1}{2} \sqrt{4a^2 + b^2} + \frac{1}{2} b$ is $= a^2$, as appears by bare Inspection: the Sum of any two Quantities drawn into their Difference being equal to the Difference of their Squares.

From hence arises the following Construction. Bisect $BC (b)$ in G , and draw AG : in BC continu-

continued out, make $GV = AG (\frac{1}{2} \sqrt{4a^2 + b^2})$
and V will be the Vertex of the Cone.

PROB. XI.

120. **T**O find the Duration of a Pendulum oscillating in the Curve of the Cycloid.

Let the Diameter of the generating Circle, FIG. 75.
or the Altitude of the whole Cycloid be $= a$;
and let HB , the Altitude of the Point Q from
whence the Pendulum begins to fall, and describe the Arch QB , be $= b$. Also let HP
 $= z$; and so $PB = 2b - z$. Now let the
Time of the Pendulum's describing QB be $= x$,
and on HB describe the Semicircle HNB , and
draw PM , pm infinitely near one another, and
perpendicular to HB ; then will PN be $=$
 $\sqrt{2bz - zz}$, $Pp = Nq = Rm = z$, and the Velocity
in P , and so in N and $Q = \sqrt{z}$.

Consequently since the Particle of the Curve
 Mm is described by an uniform Motion, the
Time of the Description of the same, viz. $\dot{x}$
is $= \frac{Mm}{\sqrt{z}}$. But from the Nature of the Cy-

cloid $Mm : mR :: BS : BP$, and $AB : BS :: BS$
 $: BP$, from the Nature of the Circle. Whence
 $BS : BP :: \sqrt{AB} : \sqrt{PB}$, and so $Mm : mR ::$
 $\sqrt{AB} : \sqrt{PB}$; therefore $Mm = \frac{mR \times \sqrt{AB}}{\sqrt{PB}}$,

and $\dot{x} = \frac{z\sqrt{a}}{\sqrt{2bz - zz}} = \frac{2bz\sqrt{a}}{2b\sqrt{2bz - zz}}$. But

$\frac{bz}{\sqrt{2bz - zz}} = Nn$; therefore $\dot{x} = \frac{2\sqrt{a} \times Nn}{2b}$.

Now when the Fluent of $\dot{x}$, viz. x does ex-
S f press

press the Time of the Descent BD of the whole Arch of the Cycloid, the Fluent of Nn will be = the Periphery HNB ; therefore as $2b$ the Diameter of the Circle to $\frac{1}{2}$ the Circumference thereof, so is $2\sqrt{a}$ to the Time of the Pendulums describing the Arch BQ . Consequently because $2\sqrt{a} = \frac{2a}{\sqrt{a}}$ de-

notes the Time of the perpendicular Descent thro' AB , we have the following Theorem, viz. The Time of an whole Oscillation thro' any Arch of the Cycloid, is to the Time of the perpendicular Descent thro' the Diameter AB of the generating Circle, as the Periphery of a Circle to the Diameter.

C O R O L L.

HENCE the Times of describing all Arches of a Cycloid are equal.

P R O B. XII.

121. **T**HE Course and Difference of Latitude of two Places being given: to find the Difference of Longitude.

FIG. 76. Let P be the Pole; the Circle ABF , the Equator; PCB , PD Meridians; ACG the Rhumb Line passing thro' two given Places A and C . Draw $Pdcb$ infinitely near PCB , and with the Distance PC describe the Arch CD . Now make the Radius $PA = a$; the Difference of Latitude of the two Places, viz. AD or $BC = y$; the Difference of Longitude sought $AB = x$; the Tangent of the given Course (or constant Angle that the Rhumb Line makes with

with any Meridian) viz. of the Angle $Ccd = m$; also make the Sine of the Latitude of the Place $C = r$, and the Cosine $= z$. All these are variable Quantities except $AP (a)$, and the Tangent of the given Course $= m$.

Now since the Arches Bb, Cd are similar; therefore $PB (a) : \text{Cosine of Lat. of } C (z) ::$

$$Bb(\dot{x}) : Cd = \frac{z\dot{x}}{a}. \text{ Again, from the Nature of}$$

the Circle, PB^2 — Square of Sine Lat. of C , viz. $a^2 - r^2$ is $= z^2$, and $cd(\dot{y})$ is $= \sqrt{r^2 + \dot{z}^2}$, as easily appears. Whence throwing the E-

quation $a^2 - r^2 = z^2$ into Fluxions, and we get $\dot{z}^2 = \frac{r^2 \dot{r}^2}{a^2 - r^2}$. And so putting $\frac{r^2 \dot{r}^2}{a^2 - r^2}$

for $\dot{z}^2$ in the Equation $\dot{y} = \sqrt{r^2 + \dot{z}^2}$, and

there arises $\dot{y} = \frac{ar}{z}$. If $cd(\dot{y})$ be made the Ra-

dius, then will Cd be the Tangent of the Angle Ccd of the Course $= m$; therefore $PB(a)$

$$: m :: cd(\dot{y}) : Cd = \frac{z\dot{x}}{a}; \text{ and drawing the}$$

Means and Extremes into each other, there a-

ries $m\dot{y} = z\dot{x}$; and so $\dot{x} = \frac{m\dot{y}}{z}$. Whence sub-

stituting $\frac{ar}{z}$ for $\dot{y}$, and we have $\dot{x} = \frac{mar}{z^2} =$

$\frac{mar}{a^2 - r^2}$; and finding the Fluents, there arises

$$BA(x) = m \times \frac{r}{a} + \frac{r^3}{3a^3} + \frac{r^5}{5a^5} + \frac{r^7}{7a^7}, \text{ \&c.} =$$

Difference of Longitude of the Places A and C .

The Fluent of $\frac{amar}{a^2 - r^2}$ may be had likewise

after Mr. Cotes's way in the Measure of a Ra-

S f 2 tio;

tio; for it may be referred to the second Form of his Tables; whence making $\theta = 0$, $n = 2$,

$$d = am, e = a^2, f = -1, R (= \sqrt{\frac{-e}{f}}) = a,$$

$$T \left(x^{\frac{1}{2}n} \right) = r, \text{ and } S (= \sqrt{\frac{e+fz^n}{f}}) = \sqrt{aa-rv},$$

the Fluxion of the Form $\frac{d \dot{z} z^{\theta n + \frac{1}{2}n - 1}}{e + fz^n}$ will

become $\frac{amr}{a^2 - r^2}$, and the Fluent $\frac{2}{ae} dR \left| \frac{R+T}{S} \right|$

will be $m \left| \frac{a+r}{\sqrt{a^2 - r^2}} \right| = m \left| \frac{a+r}{z} \right|$ being the Ex-

pression for the Difference of Longitude BA .

That is, in Words, the Difference of Longitude is equal to the Measure of the Ratio of the Radius added to the Sine of the Latitude of the Place C , and the Sine Complement of the same, the Tangent of the Course being the Module; and if E be some other Place in the same Rhumb Line, the Sine of whose Latitude is given; then by the same Rule we can get the Difference of Longitude AH , and so the Difference of Longitude BH of the Places C and E .

C O R O L.

IF the Rhumb Line AC be $= u$, and cd (y) be made the Radius; then will Cc be the Secant of the Course. Whence PB (a): Cc : $:cd$ (y): Cc (u); and so $au = Cc \times y$, and taking the Fluents $au = Cc \times y$, therefore $u = \frac{Cc \times y}{a}$; that is, as the Radius is to the Secant of the Course, so is the Difference of Latitude AD to the Length AC (y).

P R O B.

PROB. XIII.

122. **TO** cube the Solids generated by the Rotation of the Conchoidal Spaces $CPGB$, and $BGQc$ about the Line ABC drawn from the Pole A at right Angles to the Asymptote BG . FIG. 77.

Draw Ap infinitely near AP ; from A describe the small Arches Qs , Gr , Pn , with the Distance BC or Bc , from A describe the Arch EF , and from F , Q , P draw FH , QK , PI perpendicular to AE . Call $cB (= AE = AF = BC = QG = GB)$ a , AB , b , EH , x , AH , z , and HF , y . Now from the Similarity of the Triangles ABG , AHF , we have $AH(z):AF(a)::AB(b):AG = \frac{ab}{z}$. Whence $AQ = \frac{ab}{z}$

$-a$, and for the same Reason $AP = \frac{ab}{z} + a$.

Again, $AF(a):FH(y)::AQ\left(\frac{ab}{z} - a\right):QK = \frac{by}{z} - y$. So likewise $PI = \frac{by}{z} + y$. Now

the Fluxion of the Arch EF , viz. Ff will be $\frac{az}{y}$; and since the Sectors AFf , AQs are si-

milar, therefore $AF(a):Ff\left(\frac{az}{y}\right)::AQ$

$\left(\frac{ab}{z} - a\right):Qs = \frac{abz}{zy} - az$. In like manner

$Pn = \frac{abz}{zy} + az$.

Again, the Fluxion of the Solid generated by the Space AQ during the Motion above-mention'd, will be $= \frac{1}{3} AQ \times Qs$ drawn into the

the Periphery described by the Point Q . So also will the Fluxion of the Solid generated by the Space ACP be $= \frac{1}{3} AP \times Pn$ into the Periphery described by the Point P , and the Fluxion of the Cone generated by the right-angled Triangle ABG during the aforesaid Motion will be $\frac{1}{3} AG \times Gr$ into the Periphery described by the Point G ; therefore the Fluxion of the Solid generated from AcQ will be

$$\frac{ab}{3z} - \frac{a}{3} \times \frac{p}{r} \times \frac{by}{z} - y \times \frac{abz}{zy} - a \dot{z} \left(\frac{p}{r} \text{ being the Ratio of the Radius to the Periphery of a} \right.$$

$$\text{Circle}) = \frac{p}{r} \times \frac{a^2 b^3 \dot{z}}{3z^3} - \frac{a^2 b^2 \dot{z}}{z^2} + \frac{a^2 b \dot{z}}{z} - \frac{a^2 \dot{z}}{3}; \text{ and}$$

the Fluxion of the Solid generated from ACP

$$\text{will be } \frac{ab}{3z} + \frac{a}{3} \times \frac{p}{r} \times \frac{by}{z} + y \times \frac{by}{z} + y = \frac{p}{r} \times \frac{a^2 b^3 \dot{z}}{3z^3} + \frac{a^2 b^2 \dot{z}}{z^2} + \frac{a^2 b \dot{z}}{z} + \frac{a^2 \dot{z}}{3}; \text{ and the Difference of}$$

these two last mentioned Fluxions will be $=$

$$\frac{p}{r} \times \frac{2a^2 b^3 \dot{z}}{3z^3} + \frac{2}{3} a^2 \dot{z} = \text{Fluxion of the Sum of}$$

the two Solids generated by the Spaces $cQGB$,

$$BGP C; \text{ and so } \frac{p}{r} \times \frac{a^2 b^3 \dot{z}}{3z^3} + \frac{1}{3} a^2 \dot{z} \text{ is the Flu-}$$

xion of $\frac{1}{2}$ the Sum; the Fluent of which will

$$\text{be } \frac{p}{r} \times \frac{a^2 b^3}{z} + \frac{1}{3} a^2 z, = (\text{substituting } a-x \text{ for } z)$$

$$\frac{p}{r} \times \frac{a^2 b^3}{a-x} - \frac{a^3 - a^2 x}{3} = \frac{p}{r} \times \frac{a^2 b^3 - \frac{a^4}{3} + \frac{2a^3 x}{3} - \frac{a^2 x^2}{3}}{a-x}$$

and making $x = 0$, this Fluent becomes

$$\frac{p}{r} \times \frac{a^2 b^3 - a^3}{3} \text{ to be subtracted: so that the true}$$

Fluent

$$\begin{aligned} \text{Fluent is } \frac{p}{r} \times \frac{a^2 b^2 - \frac{a^4}{3} + \frac{2a^2 x}{3} - \frac{a^2 x^2}{3}}{a-x} - ab^2 + a^3 \\ = \frac{p}{r} \times \frac{ab^2 x + \frac{a^3 x}{3} - \frac{a^2 x^2}{3}}{a-x}. \end{aligned}$$

Much after the same way as the Fluxion of $\frac{1}{2}$ the Sum of the Solids was found, we may

get the Fluxion of $\frac{1}{2}$ the Difference: for from the similar Triangles AHF , ABG , there arises $AH(z):AF(a)::AB(b):AG=\frac{ab}{z}$, and

$$AH(z):HF(y)::AB(b):BG=\frac{by}{z}; \text{ and since}$$

the Sectors AFf , APr are also similar, therefore $AF(a):Ff\left(\frac{az}{y}\right)::AG\left(\frac{ab}{z}\right):Gr=\frac{abz}{yz}$;

and consequently the Fluxion of the Cone described by the right-angled Triangle ABG will be $=\frac{p}{r} \times \frac{ab}{3z} \times \frac{by}{z} \times \frac{abz}{yz} = \frac{p}{r} \times \frac{a^2 b^3 \dot{z}}{3z^3}$, from

which taking the Fluxion (before found) of the Solid described by the Space AcQ , and the Remainder $\frac{p}{r} \times \frac{a^2 b^2 \dot{z}}{z^2} - \frac{a^2 b \dot{z}}{z} + \frac{a^2 \dot{z}}{3}$ is the Fluxion of the Solid generated by the Space $cQGB$. And if $\frac{p}{r} \times \frac{a^2 b^3 \dot{z}}{3z^3}$ be taken from the

Fluxion of the Solid described by the Space ACP , the Remainder $\frac{p}{r} \times \frac{a^2 b^2 \dot{z}}{z^2} + \frac{a^2 b \dot{z}}{z} + \frac{a^2 \dot{z}}{3}$ is the Fluxion of the other Solid $BGPG$.
from

From which subtracting the Fluxion of the Solid $cQGB$, and the Remainder $\frac{p}{r} \times \frac{z a^2 b \dot{z}}{z}$ is the Difference of the said Solids; so that $\frac{p}{r} \times \frac{a^2 b \dot{z}}{z}$ is the Fluxion of $\frac{1}{2}$ the Difference of the Solids. Which may be compar'd with that of the first Form in the Tables of Mr. *Gotes*, (having first substituted $a-x$ for z , and $-x$ for $\dot{z}$, the Fluxion becoming $\frac{p}{r} \times \frac{1 a^2 b \dot{x}}{a-x}$).

For making $z=x$, $\theta=1$, $n=1$, $d=a^2 b$, $e=a$, $f=-1$, the Fluxion $\frac{d \dot{z} z^{\theta n-1}}{e+fz^n}$ will be $= \frac{p}{r} \times \frac{a^2 b \dot{x}}{a-x}$, and the Fluent $\frac{d}{nf} \left| \frac{e+fz^n}{e} \right|$ becomes $= -\frac{p}{r} a a b \left| \frac{a-x}{a} \right| = -\frac{p}{r} a a b \left| \frac{a}{a-x} \right|$; since the Logarithm of the Ratio of a to $a-x$ with an affirmative Sign, is equal to the Logarithm of the Ratio of $a-x$ to a with a negative Sign.

The Fluents or Quantities of $\frac{1}{2}$ the Sum and $\frac{1}{2}$ the Difference of the Solids being thus found we proceed next to their Construction, beginning with that of the half Sum. From what has been already said, it is easy to find the Fluxion of the Sector of the Sphere described by the circular Sector AEF , and so the Fluent or Quantity of that Sector; the Fluxion being $= \frac{p}{r} \times \frac{a}{3} \times y \times \frac{a \dot{z}}{y} = \frac{p}{r} \times \frac{a^2 \dot{z}}{3}$, and the Fluent $= \frac{p}{r} \times \frac{a^2 z}{3} = \frac{p}{r} \times \frac{a^3 - a^2 x}{3}$, by substituting $a-x$ for z ; and making $x=0$, the said Fluent will become $\frac{p}{r} \times \frac{a^3}{3}$, to be subtracted from

the

the other. Whence the true Fluent is $\frac{p}{r} \times \frac{-a^2x}{3}$ or $\frac{p}{r} \times \frac{a^2x}{3} =$ Solidity of the Sector of the Sphere described as aforefaid.

This being granted, make $\overline{AE}^2 (aa) : 3 AB \times AG + \overline{AE}^2 (3b \times \frac{ab}{a-x} + aa) ::$ Sector Sphe.

$\frac{p}{r} \times \frac{a^2x}{3} : \text{Solidity of } \frac{1}{2} \text{ the Sum of the Solids}$
 $= \frac{p}{r} \times ab^2x + \frac{a^3x}{3} - \frac{a^2x^2}{3}$ being the same Expression as that before found.

Lastly, To construct the Expression $\frac{p}{r} aab \left| \frac{a}{a-x} \right|$, we know that $\frac{p}{2r} aa$ is the Area of a Circle, having $BC(a)$ for a Radius; which drawn into $b \left| \frac{a}{a-x} \right|$ will be the Solidity of $\frac{1}{2}$ the

Difference. But because of the similar Triangles AHF , ABG , the Ratio of $a = AF$ to $a-x = AH$, is equal to the Ratio of AG to AB ; therefore the Value of $\frac{1}{2}$ the Difference of the Solids to be cubed is equal to a Cylinder, the Diameter of whose Base is $cC(2a)$; and Altitude the Measure of the duplicate Ratio of AG to AB , the Module being $AB(b)$.

for $\frac{p}{2r} aab \left| \frac{a^2}{a-x} \right|$ is $= \frac{p}{r} aab \left| \frac{a}{a-x} \right|$; because the

Logarithm of any Ratio doubled is that of the duplicate Ratio.

PROB. XIV.

FIG. 78. 123. **T**O find the Nature of a Curve *AMC* being such, that if a Vessel be described by the Revolution of the same about the Axis *AB* perpendicular to the Horizon; and then filled with Water, which afterwards runs out of a small round Hole in the Bottom *A*, the Surface of the Water shall descend equal Spaces in equal Times; admitting the Velocity of the Water running out, to be as the square Root of the Altitude of its Surface above the Hole.

Let $AB=a$, $AP=x$ be any indeterminate Altitude of the Water, and let $PM=y$. Also let the Surface of the Hole be $=b$, the Velocity $=v$, and the Time of Descent of the Surface $=t$. Which from the Condition of the Problem is as $a-x$, but $\frac{\dot{x}}{t}$ is as v ; that is, as $\sqrt{x}$ or as $\sqrt{ax}$, from the Condition of the Problem; therefore $\frac{\dot{x}}{t} : \sqrt{ax} :: \text{Surf. Hole } b : \text{Surf. Water } y^2$. But since t may be taken $= a-x$, therefore $t=\dot{x}$; and so $1 : \sqrt{ax} :: b^2 : y^2$. Whence $b\sqrt{ax}=y^2$, and $ab^2x=y^4$. Consequently the Curve *ACM* is a biquadratick Parabola.

PROB. XV.

FIG. 79. 124. **I**F *AC* be a horizontal Line upon the Point *C* of which stands an upright Parallelepipedon *CD*; one of the plain Surfaces of which is perpendicular to *AC*; it is required to find the Angle *CAB*, in the Point *A* of which one End *A* of a long Solid

Solid AB being set, so as with its other End B it may bear against the Parallelepipedon CD ; the same shall press perpendicularly against CD with a greater Force than if it was set at any other Point besides A .

Let F be the Centre of Gravity of AB . From F draw FE perpendicular to AC , and from E , EI perpendicular to AB . Also from B , BH perpendicular to AB , and $=AE$; and from H , HG perpendicular to BD . Make $AB=a$, $AF=b$, and $CB=x$.

Now the Pressure of the Solid AB in the Point B , and Direction HB against CD will be as AE , and so may be represented by it; therefore since $BH=AE$, the direct Pressure against CD in the Point B will be $=HG$. This is shewn in the Principles of Mechanicks. Whence $CB \times HG$ is the Effect of the perpendicular Pressure of the Solid AB against CD in the Point B , which must be a *Maximum*. Again, $AB(a) : AC(\sqrt{a^2-x^2}) :: AF$

$(b) : AE = \frac{b}{a} \sqrt{a^2-x^2}$. And since the Triangles AEI , ACB are similar, therefore $AB(a) : BC(x) :: AE\left(\frac{b}{a} \sqrt{a^2-x^2}\right) : EI = HG = \frac{bx}{a^2} \sqrt{a^2-x^2}$. Whence $CB \times HG = \frac{bx^2}{a^2} \sqrt{a^2-x^2}$.

The Fluxion $\frac{4a^2b^2x^3\dot{x} - 6b^2x^5\dot{x}}{2a^4x \times \frac{bx^2}{a^2} \sqrt{a^2-x^2}}$ of which must be $=0$; therefore $2a^2=3x^2$, and so $x = \sqrt{\frac{2aa}{3}}$.

Whence as AB to $\sqrt{\frac{2}{3}} AB$, so is the Radius to the Sine of the Angle $CAB = 54^\circ 44'$.

PROB. XVI.

FIG. 80.

125. **T**O find the Nature of a Curve *ACEDB*, along which if a Body freely falls by its own Weight from the given Point *A* to the given Point *B*; the Time of the Descent shall be less than the Time of the Descent of a Body from *A* to *B* along any other Curve passing thro' the given Points *A* and *B*.

Let *C* and *D* be two given Points in the Curve infinitely near each other. Also let the intermediate Point *E* of the Curve be taken such, that drawing *DI* and *EH* perpendicular, and *AK*, *CL*, *EM* parallel to the Horizon, the small Line *EL* may be $= DM$.

Now since *ACEDB* is supposed to be the Curve along which the Body falls from *A* to *B* in the least Time possible; it will fall from *C* to *D*, any Part of the said Curve, in the least Time possible. For if it does not, suppose it to fall along *CGD* in a less Time than along *CED*; then the Curve *ACGDB* will be described in a less Time than the Curve *ACEDB*. Which is absurd.

Because the Points *C* and *D* are given in Position, therefore the Lines *HE*, *ID*, $EL = DM$, are all given or invariable; and *CL*, *CE*; *EM*, *ED* are variable. Make $EL = DM = m$, $HE = b$, $ID = p$; also $CL = u$, $EM = z$; then the Time of the Description of the small Line *CE* (being as *CE* directly, and the Velocity inversly) will be as $\frac{CE}{\sqrt{HE}}$

$\left(\sqrt{\frac{m^2 + n^2}{b}}\right)$, the Velocity during the Descrip-

ption

ption of infinitely small right Lines being looked upon as equable; so likewise the Time of the Body's describing the short Line ED is

as $\frac{ED}{\sqrt{TD}} \left(\sqrt{\frac{m^2 + z^2}{p}} \right)$. Consequently the Sum of these Times must be a *Minimum*, viz.

$$\sqrt{\frac{m^2 + u^2}{b}} + \sqrt{\frac{m^2 + z^2}{p}};$$

and so the Fluxion of the same, which is $\frac{u \dot{u}}{b^{\frac{1}{2}} \sqrt{m^2 + u^2}} + \frac{z \dot{z}}{p^{\frac{1}{2}} \sqrt{m^2 + z^2}}$ must be $= 0$, but $CN = u + z$ is invariable; whence $\dot{u} + \dot{z} = 0$, and so $\dot{u} = -\dot{z}$; therefore

$$\frac{u}{b^{\frac{1}{2}} \sqrt{m^2 + u^2}} = \frac{z}{p^{\frac{1}{2}} \sqrt{m^2 + z^2}}.$$

Now if $AH = x$, and $HE = y$; then will $EM = \dot{x}$, and $MD = \dot{y}$; and so $ED = \sqrt{\dot{x}^2 + \dot{y}^2}$. Consequently the Fluxion of the Curve is always as $\frac{\dot{x}}{y^{\frac{1}{2}}}$; that is, in the direct

Ratio of the Fluxion of the Absciss $AH(x)$, and the reciprocal subduplicate Ratio of the correspondent Ordinate $HE(y)$.

Now a Curve that has this Property will be found to be a Cycloid, passing through the given Points A and B , with the Vertex downwards, as may be easily shewn thus: Suppose A and B to be so posited, that ACD be a Semi-cycloid, &c.

Describe the generating Semicircle KPB . Continue out EM to P and Q . Draw the Tangent $CE\dot{T}$ to the Curve in E , and from B draw the Chord BP . Let $BK = a$; then will $KQ = y$, and $BQ = a - y$. Now it is a noted Property of the Cycloid for the Tangent ET to be parallel to the Chord BP

of

of the correspondent Arch of the generating Circle. Whence the Triangles BPQ , and ECL are similar; therefore $PQ (\sqrt{ay - y^2}) : BP (\sqrt{a^2 - y^2}) :: CL (\dot{x}) : CE = \frac{\dot{x} \sqrt{a^2 - ay}}{\sqrt{ay - y^2}} =$

$$\frac{\dot{x} \times \sqrt{a} \times \sqrt{a - y}}{\sqrt{y} \times \sqrt{a - y}} = \frac{\dot{x} \times \sqrt{a}}{\sqrt{y}}. \text{ Which Expression}$$

is as $\frac{\dot{x}}{y^{\frac{1}{2}}}$, because $\sqrt{a}$ is given or invariable; therefore the Cycloid $ACEDB$ is the Curve of the swiftest Descent.

PROB. XVII.

FIG. 82. 126. **T**O find the Nature of a Curve DM being such, that the Solid described by the Revolution of it about the Axis AP , moving in a Fluid (such as that in Art. 119.) in the Direction of the Axis, shall be less resisted than any other Superficies described by what Curve soever terminating in the given Points D and M about the same Axis AP , and moving in the same manner.

FIG. 81. Let the right Lines MN , NO be supposed to be two infinitely small Parts of the Curve sought. Now the Surfaces described by these will meet with a less Resistance than the two Superficies described by any two Parts drawn from the Points O and M to any Point besides N . This is very plain: for if the same be denied, the Consequence will be that N is not in the Curve sought.

This being granted; from the Points M , N , O , draw MP , NQ , OH perpendicular to the Axis; and from M draw MF parallel to the same; also thro' N draw another indefinite Parallel

Parallel GNT cutting OH in G . Now the Resistance of the little Superficies described by MN , is to the Resistance of the little *Annulus* described by FN , as the Square of the Sine of the Angle of Incidence to the Square of the Radius; that is, (because the Angle FMN is = Angle of Incidence TNM) making MN the Radius, as $\overline{FN}^2$ to $\overline{MN}^2$. And if the Resistance of the *Annulus*, described by FN , be represented by itself; or rather by $\overline{QF}^2 + 2\overline{QF} \times FN + \overline{FN}^2$ minus $\overline{QF}^2$ (being the Difference of $\overline{QN}^2$ and $\overline{QF}^2$) which Remainder is $2\overline{QF} \times FN + \overline{FN}^2$; this Difference being always as that *Annulus*. But because FN is infinitely less than QF , therefore $\overline{FN}^2$ is infinitely less than $2\overline{QF} \times FN$; and so it may be rejected: And the Resistance of the *Annulus* aforesaid will be $2\overline{QF} \times FN$; or $\overline{QF} \times FN$; or $PM \times FN$. Whence at length making $\overline{MN}^2 : \overline{NF}^2 :: PM \times FN : \frac{NF^3 \times PM}{\overline{MN}^2}$. And

this will express the Resistance of the Superficies described by MN ; so will $\frac{GO^3 \times \overline{QN}}{\overline{ON}^3}$ be that described by ON .

Again, let the Points O , M , and the right Line GT be given in Position; then we are to enquire into the Situation of MN , NO , being such that the Resistance of the Superficies described by them, be less than the Superficies described by any other Lines On , nM . In order to this, let $PM = a$, $FN = b$, $GO = c$, $NQ = e$. These are all given, and invariable; and

and let the variable Quantities MN be $= x$,
and $NO = z$.

Then will the Resistance of the Superficies
described by MN be $= \frac{ab^3}{x^2}$, as has been shewn

above. In like manner $\frac{ec^3}{z^2}$ will be the Resistance
of the Superficies described by ON . Whence
 $\frac{ab^3}{x^2} + \frac{ec^3}{z^2}$ must be a *Minimum*; that is, $\frac{ab^3 x \dot{x}}{x^4}$
 $+ \frac{ec^3 z \dot{z}}{z^3}$, or $\frac{ab^3 \dot{x}}{x^3} + \frac{ec^3 \dot{z}}{z^3} = 0$. Whence $-\frac{ab^3 \dot{x}}{x^3}$
 $= \frac{ec^3 \dot{z}}{z^3}$.

Now in order to get an affirmative Value of
 $-\dot{x}$, affected with $\dot{z}$; assume the Point n in-
finitely near N , and draw the right Lines On ,
 Mn , to which draw the Perpendiculars NS ,
 NR . Then since Nn , and consequently NS
is infinitely less than NM ; therefore $MN =$
 MR , and so the Angle $MNR =$ Angle MRN
 $=$ right Angle. Whence the Angle $R N n$ is
 $=$ Angle $F N M$; for either of them added to
the Angle $M N n$ makes a right Angle. Like-
wise since the Triangles NSn , nGO are equi-
angular, they having each a right Angle OGn ,
 NSn , and the Angle ONG common; there-
fore the Angle SNn is $=$ Angle $SOG =$ An-
gle NOG ; since the Angle NOS is infinitely
small. Therefore making Nn the Radius, we
have $Rn (-\dot{x}) : Sn (\dot{z}) :: \text{Sine Ang. } FNM :$
 $\text{Sine Ang. } GON$. Supposing MN the Radi-
us, (that is, making $MF = m$, and $NG = n$,
and assuming $NL = MN$, and drawing LK
parallel to OG) as $MF(m) : NG(n)$; therefore
 $-\dot{x} = \frac{mz\dot{z}}{nx}$; which being put in the Equati-

on

on above, and we get $\frac{ab^3mz\dot{z}}{nx^4} = \frac{ec^3\dot{z}}{z^2}$. Whence

$$\frac{ab^3m}{x^4} = \frac{ec^3n}{z^4}.$$

Now draw AB (a) perpendicular to the Axis AP , and the right Lines BC , BE parallel to the two infinitely small Subtenses MN , NO ; then will $4\overline{AB}^2 \times AC$ ($4a^2 \times \frac{am}{b}$): $\overline{BC}^2$

$\left(\frac{a^3x^3}{b^3}\right) :: BC \left(\frac{ax}{b}\right) : MP$ (a); and multiply-

ing the Means and Extremes, we have $\frac{b^3m}{x^4}$

$= \frac{1}{4}$, or $\frac{ab^3m}{x^4} = \frac{a}{4}$. In like manner, $4\overline{AB}^2$

$\times AE : \overline{BE}^2 : BE : NQ = \frac{ec^3n}{z^4}$.

Therefore the Nature of the Curve DM is such, that if AB be taken in the Line AK perpendicular to the Axis $= a$; and drawing BC parallel to a Tangent to the Curve in any Point M , we have always $4\overline{AB}^2 \times AC : \overline{BC}^3 : BC : MP$ the Ordinate.

Now this Curve may be described by the Logarithmick Curve thus: Assume $AB = a$ in AK ; and in AP continued out towards A , take $AE = \sqrt[4]{\frac{1}{3}}aa$; and thro' the Point E describe the Logarithmick Curve FEN to the Asymptote AK , whose Subtangent is $= \frac{1}{4}a$. Then assuming AC (suppose $= s$) at pleasure, and drawing CN parallel to AK , take $AK =$

$$\frac{aa}{4s} + \frac{1}{2}s + \frac{s^3}{4aa}, \text{ and } AP = \frac{ss}{4a} + \frac{3s^4}{16a^3} - \frac{s}{48}a$$

$\pm CN$, viz. — when AC is greater than AE , and $+$ when it is less; and draw the

U u

right

right Lines KM , PM parallel to AP , AK ; then will their Point of Intersection M be in the Curve DM sought.

For making $AP = x$, $PM = y$, and $AC = s$; the Property the Curve must have, gives AK or $PM(y) = \frac{a^4 + 2aass + s^4}{4aas}$, and con-

sequently $y = \frac{1}{2}s + \frac{3ss}{4aa} - \frac{aas}{4s}$. Then since BC

is parallel to the Tangent in M . Therefore $\dot{x} = \frac{s\dot{y}}{a} = \frac{s\dot{s}}{2a} + \frac{3s^3\dot{s}}{4a^3} - \frac{a\dot{s}}{4s}$, the Fluent of which is

$AP(x) = \frac{s\dot{s}}{4a} + \frac{3s^4}{16a^3}$, minus the Fluent of $\frac{a\dot{s}}{4s}$ plus or minus some invariable Quantity. This

Quantity we will take to be $\frac{5}{48}a$, which we

subtracted, that so CN , which from the Nature of the Logarithmick Curve FEN is the

Fluent of $\pm \frac{a\dot{s}}{4s}$ becoming $= 0$, $AP(x)$ may be $= 0$ also. Whence, $\&c$.

When $AC = AE$, the Ordinate PM , which is then a *Minimum*, becomes $AD = \frac{4}{3}AE$, and the Tangent in D will be parallel to BE . But if AC be taken less than AE , the Part DO of the Curve will be describ'd convex towards DM , diverging more and more from AP , AK . Therefore the Solid of the least Resistance may be convex or concave, or partly convex and partly concave, and the Point D is a Point of *Retrogression*.

Note, The Logarithmick Curve may be describ'd easily after this manner. For you need only take $CN =$ to the Measure of the Ratio between AE and AC , the Module being $\frac{1}{4}AB(a)$.

PROB.

PROB. XVIII.

127. **TO** find the Angle ABC , which the Plane of the Sail of a Windmill in Figure of a right-angled Parallelogram, whose given Breadth is CB , makes with the Axis AB being such, that a given Wind blowing in the Direction of the Axis, shall drive it round with a greater Force than if it had any other Inclination to the Axis.

Draw CD perpendicular to the Axis. Let FIG. 84.
 $CB = a$, and $DB = x$.

If BC represents the Number of Particles of the Air striking the Sail when it is perpendicular to the Axis AB ; then will DC be the Number of Particles striking, when the Sail is inclined to the Axis in the Angle DBC .

Now it easily follows from the Principles of Mechanicks, that the Force of the Sail in the Direction DC will be as $BD \times DC \times DC$

$= BD \times \overline{DC}^2$; that is, the Sail will be drove round by the given Wind with

a Force that is always as $BD \times \overline{DC}^2$, which consequently must be a *Maximum*. But

$BD \times \overline{DC}^2$ is $= x \times aa - xx = a^2x - x^3$. Whence $a^2x - 3x^2x = 0$, and $a^2 = 3x^2$; therefore $\frac{1}{3}a^2 = x^2$, and so $\sqrt{\frac{1}{3}a^2} = x$. Whence as CB to $\sqrt{\frac{1}{3}a^2}$, so is the Radius to the Sine of the Angle C ; the Complement of which is the Angle ABC sought. Consequently the Angle C is $35^\circ. 16'$. and the Angle ABC , $54^\circ. 44'$.

SCHOLIUM.

THIS Problem, on account of its easy Solution, should have been antecedent to some of the others, especially the two last Problems; but before I thought of it, all the others were printed off: yet rather than leave it out, I have added the same here. *Monsieur Mariot*, in his *Hydrostaticks*, has endeavour'd at a Solution of it; but his Conclusion is wrong. *Monsieur Parent* has done it, as we find in the *History of the Royal Academy of Sciences at Paris*; but we have not his Process, neither in the *History* or *Memoirs*, as I can find.

F I N I S.

10 MA 65





ERRATA in the APPENDIX.

PAGE 3. line 14. for *denominates* r. *Denominators*. P. 5. l. 1. for *b* r. *x*. P. 7. l. 11 and 14. for *Surd* r. *Series*. P. 10. l. 3. for P_n^m r. P_n^m . Ibid. l. 7. for $A (= P_n^m) a^5$ r. $A (= P_n^m) = a^5$. P. 23. l. 21. for *will be* xy , r. *will be* xy . P. 24. l. 27. for *a* r. *an*. P. 48. l. 11. for $= \frac{ax + xb}{a}$ r. $= \frac{2ax + xb}{a}$. P. 57. l. 11, 12. for $\frac{b}{a} \sqrt{aa - aa}$, and $\frac{b}{a} \dot{x} \sqrt{aa - aa}$, r. $\frac{b}{a} \sqrt{xx - aa}$, and $\frac{b}{a} \dot{x} \sqrt{xx - aa}$. P. 58. l. 1. for *Art. 11.* r. *Art. 14.* Ibid. l. 2. del. =. Ibid. l. 10. for *PH* r. *CH*. P. 59. l. 15. for *Art. 11.* r. *Art. 14.* P. 60. l. 18. for $\sqrt{x - xx}$ r. $\dot{x} \sqrt{x - xx}$. P. 84. line 7. for *z* read *z*. Page 93. line 17. for *Quadrature* read *Rectification*. P. 104. l. 21. for $\sqrt{x^2 \dot{x}^2 + y^2}$ r. $\sqrt{\dot{x}^2 + y^2}$. P. 108. l. 7. for $\sqrt{x^2 + y^2}$ r. $\sqrt{\dot{x}^2 + y^2}$. P. 114. l. 11. dele *because x begins at B, and not at A*. P. 115. l. 7. for *AP* r. *AM*. P. 118. l. 11. for $\frac{pxx - px^2 \dot{x}}{2r}$ r. $\frac{2prxx - px^2 \dot{x}}{2r}$. Ibidem, l. 14. r. $\frac{3prx^2 - px^3}{6}$. Ibid. l. 17. for $\frac{2pr^3 - 8pr^3}{6r}$ r. $\frac{12pr^3 - pr^3}{6r}$. P. 119. l. 2. for *Semidiameter* r. $\frac{1}{3}$ of the *Semidiameter*. P. 126. l. 7. for $\frac{pr}{y}$ r. $\frac{py}{r}$. Ibid. l. 13. for $\frac{pyyx}{2r}$ r. $\frac{pyyx}{2r}$. Ibid. l. 15. for

ERRATA.

$\frac{aapx}{r}$ r. $\frac{aapx}{2r}$. P. 126. l. 19. r. $\frac{bpr}{3}$. P. 127. l. 14.

for BD r. PD . P. 135. l. 22. for *and so* $\frac{n^3}{12a}$

r. *and so* $\frac{8n^3}{12a}$. Ibid. l. 23. for $\sqrt{\frac{2}{3}}n^3$ r. $\sqrt{\frac{2n^3}{3a}}$.

P. 136. l. 8. add, *when lessen'd by $\frac{1}{2}$ of a Circle, whose Radius is $=a$.* Ibid. l. 14. r. *transverse.*

Ibid. l. 15. for PM r. PA . P. 139. l. 25. for

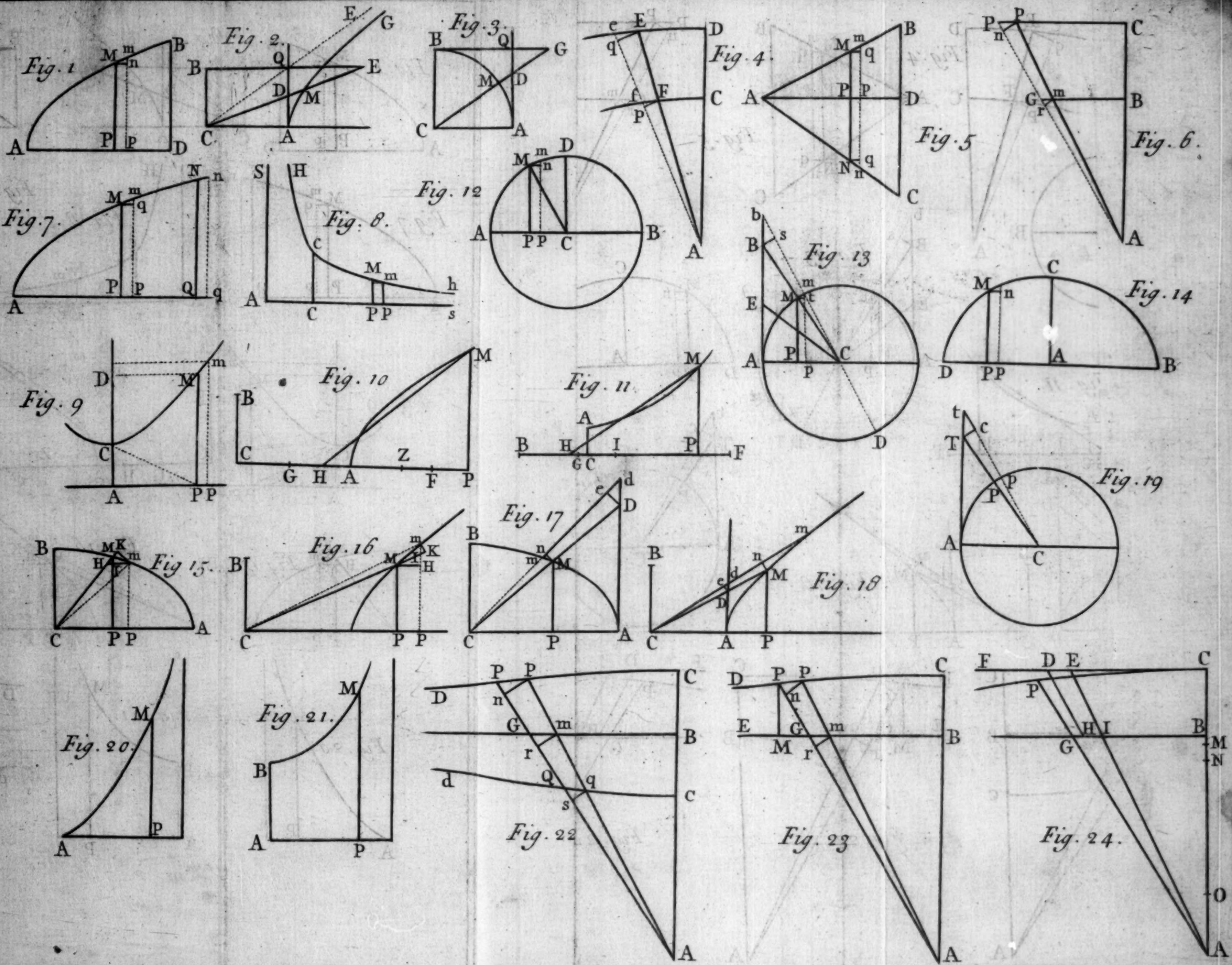
DCE r. DEC . P. 140. Art. 71. Fig. 54. wanting in the Margin. P. 141. l. 20, 21. dele, but

because y begins at the Centre C, and not at A the Vertex. P. 145. Art. 73. for Fig. 5. in the

Margin, r. Fig. 56. P. 146. l. 11, 12. dele, *since the Absciss AP (x) begins at A and not at B.*

P. 190. l. 5. for C r. B . P. 191. l. 21. dele *is.*





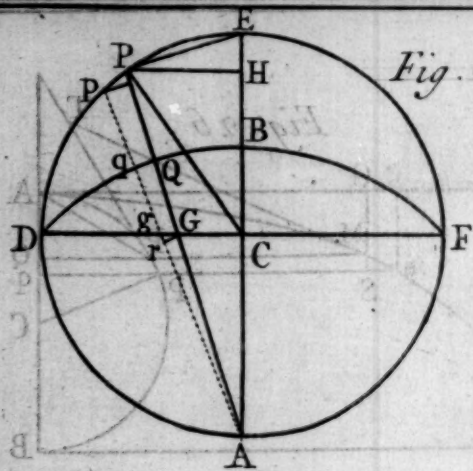


Fig. 25

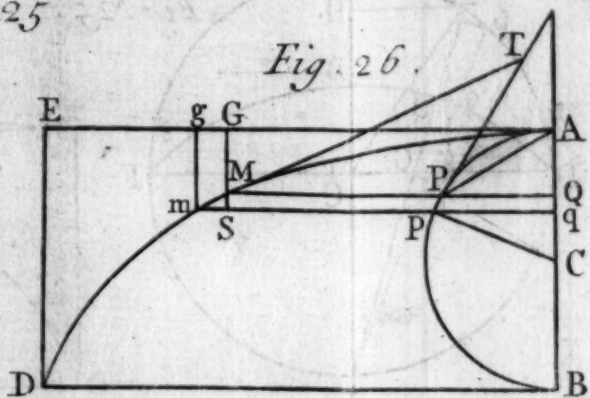


Fig. 26

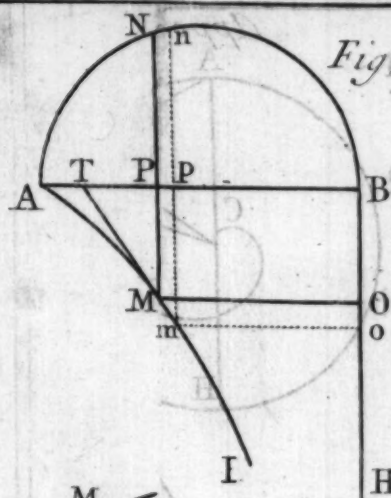


Fig. 27

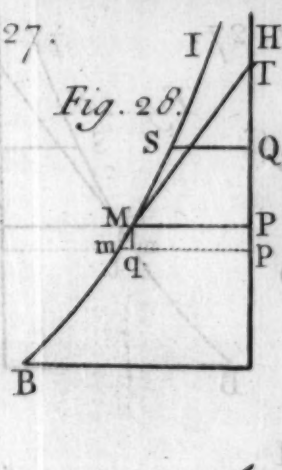


Fig. 28

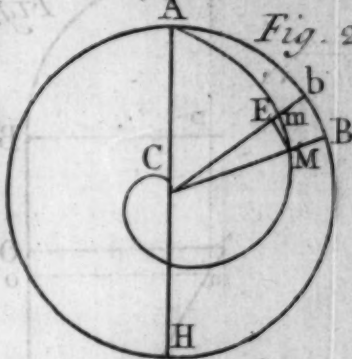


Fig. 29

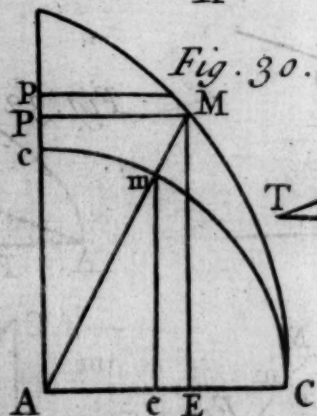


Fig. 30

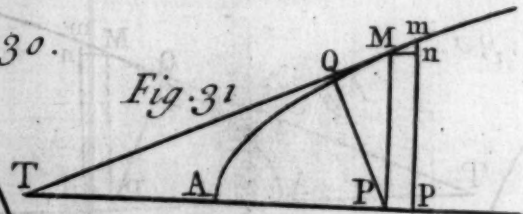


Fig. 31

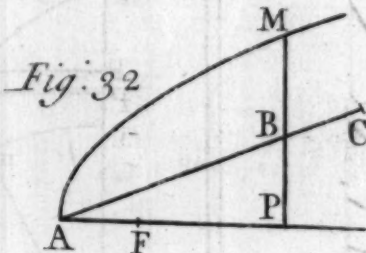


Fig. 32

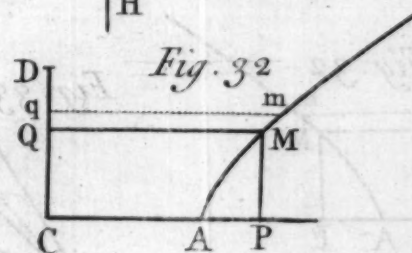


Fig. 32

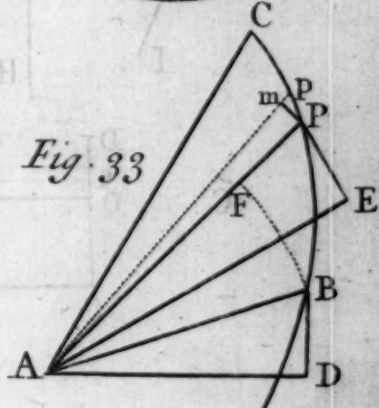


Fig. 33

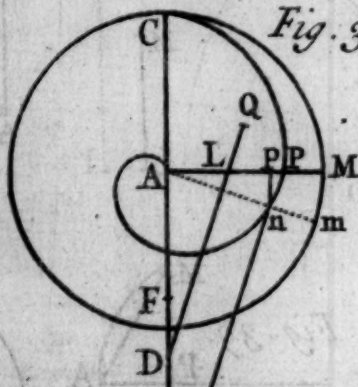


Fig. 34

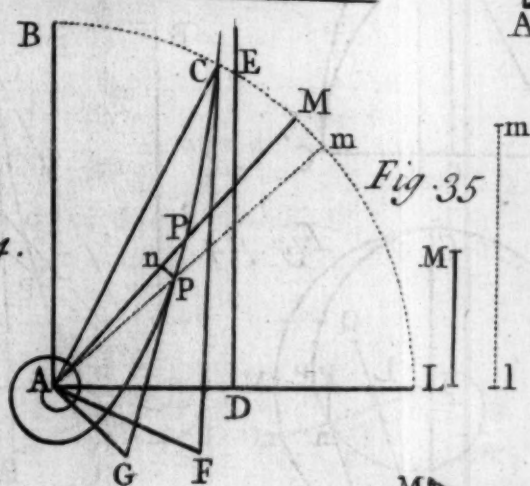


Fig. 35

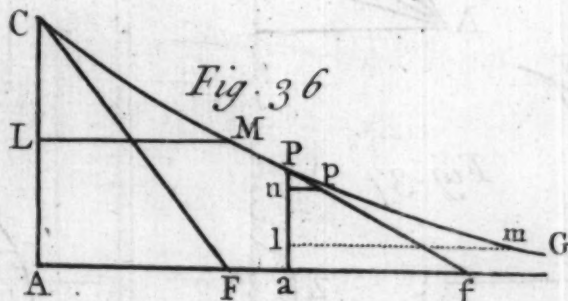


Fig. 36

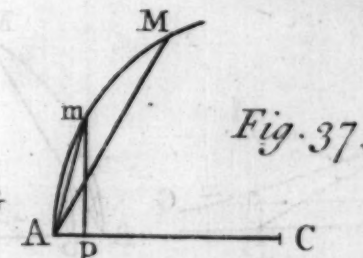


Fig. 37

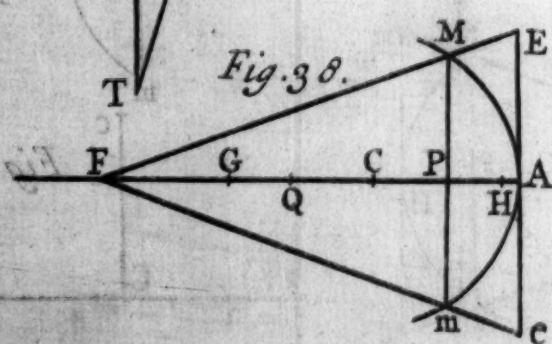


Fig. 38

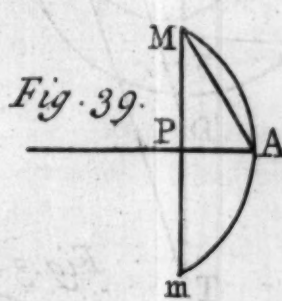


Fig. 39

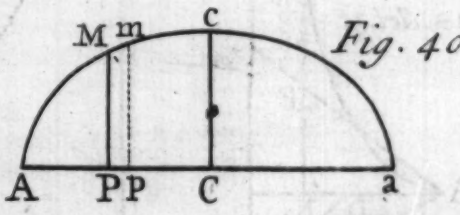


Fig. 40

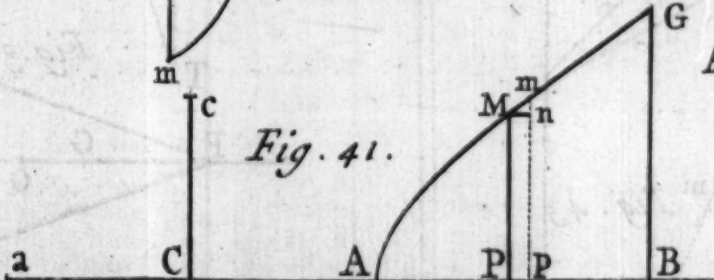


Fig. 41

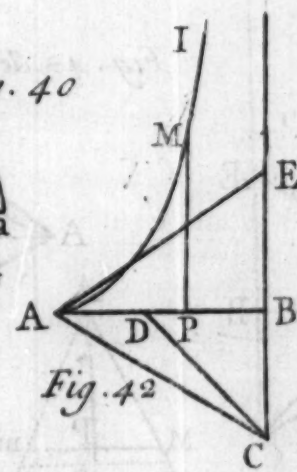


Fig. 42

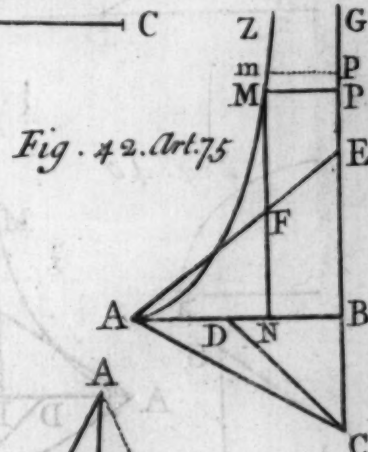


Fig. 42, Art. 75

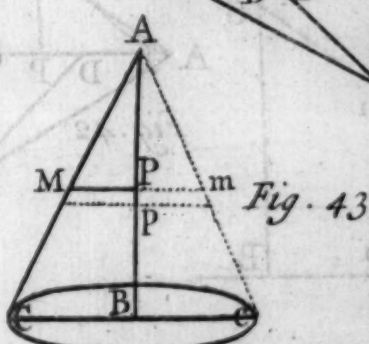


Fig. 43

